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Operators on Some Vector-Valued Orlicz Sequence Spaces

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Abstract

In this note, we give some sequences of operators which have the same function with a basis for some vector-valued Orlicz sequence spaces. Also, we characterize the space $\mathrm{B}(h_M(X),Y)$ of continuous operators from $h_M(X)$ into Y where M is an Orlicz function, X, Y are Banach spaces and $h_M(X)$ is the space of all X-valued sequences $x=(x_k)$ such that

$$\sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) < \infty \text{ for all } \rho > 0.$$

Exactly, we obtain that each $T \in \mathrm{B}(h_M(X),Y)$ is equivalent, under certain conditions, to any sequence $A=(A_k)_{k=1}^{\infty}$ of operators $A_k \in \mathrm{B}(X,Y)$.

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Bazı Vektör-Değerli Orlicz Dizi Uzayları Üzerindeki Operatörler

Özet

Bu çalışmada, bazı vektör-değerli Orlicz dizi uzayları için bir baz ile aynı işleve sahip olan bir operatör dizisi tanımladık. Ayrıca, bundan faydalanarak $h_M(X)$ uzayından Y uzayına sürekli operatörlerin $\mathrm{B}(h_M(X),Y)$ uzayını karakterize ettik. Burada M bir Orlicz fonksiyonu, X, Y Banach uzayları ve $h_M(X)$,

$$\sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) < \infty \text{ her } \rho > 0 \text{ için}$$

olacak şekildeki tüm X-değerli $x=(x_k)$ dizilerinin uzayıdır. Aslında, tam olarak, bazı şartlar altında, her bir $T \in \mathrm{B}(h_M(X),Y)$ operatörünün $A_k \in \mathrm{B}(X,Y)$ operatörlerinin bir $A=(A_k)_{k=1}^{\infty}$ dizisine denk olduğu sonucuna ulaştık.

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Anahtar Kelimeler: Fonksiyonellerin temsilleri, Orlicz dizi uzayları, Operator uzayları.

1. Introduction

After J. Lindenstrauss & L. Tzafriri [1] introduced the Orlicz sequence space ℓ_M, many variations of these spaces investigated and some of them generalized to vector-valued case. For example, a general case of vector-valued Orlicz sequence spaces is given by D. Ghosh & P. D. Srivastava [2]. In many respect, the properties of operators on some Orlicz sequence spaces (or Orlicz function spaces) are more important than structural investigation of the spaces. Especially, dealing with some integral equation and introducing an existence theorem to the equation, operators on some Orlicz spaces which has been given with respect to the equation plays a crucial role. Some related results with applications of these spaces presented in some cites such as [3,4]. Further, in the operator theory and operator algebras, characterizations of operators are useful in giving an example or a counterexample to some structural problems. In connection with the vector-valued sequence space theory, the operators from X-valued Orlicz sequence space $h_M(X)$ (or $\ell_M(X)$), where X is a Banach space, into another Banach space Y are our main interest. J. Lindenstrauss & L. Tzafriri, [5], characterized the functionals defined on ℓ_M by finding its continuous duals. Dealing with continuous duals of an abstract topological vector space X or representation of operators defined from X into another topological vector space Y, bases of the space X have an important role. This illustrated in cite [6] for some well-known scalar sequence spaces in detail. But, finding a basis for vector-valued sequence spaces is not possible, in general. In this note, we give some sequences of operators which have the same function with a basis for some vector-valued Orlicz sequence spaces. Also, by using this notion, the representation of continuous operators from $h_M(X)$ into another Banach space Y are presented.

2. Prerequisites

For some Banach space X, B_X denotes the unit sphere of X, i.e. $B_X = \{x \in X : \|x\| \le 1\}$. Specially, we use B_M instead of $B_{h_M(X)}$ in the context. Furthermore, X^* denotes the continuous dual of X, and for an operator T from X to another Banach space Y, we denote the adjoint operator of T by T^* such that $(T^* f)(x) = f(T(x))$ for all $x \in X$.

We recall, [7,1], that an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(u) > 0$ for all $u > 0$ and $M(u) \to \infty$ as $u \to \infty$. An Orlicz function M can always be represented in the following integral form:

$$M(u) = \int_0^u p(t)\,dt,$$

where p, known as the kernel of M, is right-differentiable for $t \ge 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

Consider the kernel $p(t)$ associated with Orlicz function $M(u)$, and let

$$q(s) = \sup\{t : p(t) \le s\}.$$

Then q possesses the same properties as the function p. Suppose now

$$N(v) = \int_0^v q(s)\,ds.$$

Then N is an Orlicz function. The functions M and N are called mutually complementary Orlicz functions, and they satisfy the Young inequality,

$$uv \le M(u) + N(v) \text{ for } u, v \ge 0.$$

An Orlicz function M is said to satisfy the Δ_2-condition for small u or at 0 if for each $k > 0$ there exist $R_k > 0$ and $u_k > 0$ such that $M(ku) \le R_k M(u)$, for all $u \in (0, u_k]$ [8].

The space ℓ_M consists all sequences (x_k) of scalars such that

$$\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0,$$

and it becomes a Banach space which is called an Orlicz sequence space with the Luxemburg norm

$$\|x\|_{(M)} = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\}.$$

The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(u) = u^p$, $(1 \le p < \infty)$.

Another definition of ℓ_M, [8], is given by the complementary function to M as follows:

$$\ell_M = \left\{x = (x_k) \in w : \sum_{k=1}^{\infty} x_k y_k \text{ converges, for all } y \in \widetilde{\ell}_N\right\},$$

where N is the complementary function to M, and $\widetilde{\ell}_N$ is the collection of all x in w with $\sum_{k=1}^{\infty} N(|x_k|) < \infty$. Clearly, $\widetilde{\ell}_N \subseteq \ell_N$, and ℓ_M is normed by the Orlicz norm

$$\|x\|_M = \sup\left\{\left|\sum_{k=1}^{\infty} x_k y_k\right| : \sum_{k=1}^{\infty} N(|y_k|) \le 1\right\}.$$

It was shown that these two norms on ℓ_M are equivalent.

An important closed subspace of ℓ_M is h_M, and introduced by Y. Gribanov as follows:

$$h_M = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for all } \rho > 0 \right\}.$$

Immediately, we can introduce the vector-valued extension of the spaces ℓ_M and h_M for any Banach space X. Therefore,

$$\ell_M(X) = \left\{ x \in s(X) : \sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\},$$

where $s(X)$ is the space of all X-valued sequences and $\|\cdot\|$ is the norm of X. $\ell_M(X)$ is a Banach space with the Luxemburg norm

$$\|x\|_{(M)} = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) \le 1 \right\},$$

and it coincides with ℓ_M whenever $X = \mathrm{X}$, the set of complex numbers. Further, define the closed subspace $h_M(X)$ of $\ell_M(X)$ by $x = (x_k) \in h_M(X)$ iff

$$\sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) < \infty \text{ for all } \rho > 0.$$

If M satisfies the Δ_2-condition then $h_M(X) = \ell_M(X)$.

3. Some Results on Vector-valued Orlicz Sequence Spaces

Let we begin this section with introducing another definition of $\ell_M(X)$ by the complementary function N to M.

$$\ell_M(X) = \left\{ x \in s(X) : \sum_{k=1}^{\infty} f_k(x_k) \text{ converges, for all } f = (f_k) \in \widetilde{\ell}_N(X^*) \right\},$$

where $\widetilde{\ell}_N(X^*)$ is the class of all sequences $f = (f_k)$ such that $\sum_{k=1}^{\infty} N(\|f_k\|) < \infty$.

Lemma 1. For each $x \in \ell_M(X)$,

$$\|x\|_M = \sup\left\{ \left| \sum_{k=1}^{\infty} f_k(x_k) \right| : \sum_{k=1}^{\infty} N\left(\|f_k\| \right) \le 1 \right\} < \infty$$

defines a norm on $\ell_M(X)$. This norm is said to be Orlicz norm on $\ell_M(X)$.

Lemma 2. On $\ell_M(X)$, the norms $\|\cdot\|_M$ and $\|\cdot\|_{(M)}$ are equivalent, and $\|x\|_{(M)} \le \|x\|_M \le 2\|x\|_{(M)}$.

Proofs of these lemmas easily can be given in a similar way followed in cite [8, Theorem 8.9], by using the inequality

$$\left| \sum_{k=1}^{\infty} f_k(x_k) \right| \le \sum_{k=1}^{\infty} \|f_k\| \cdot \|x_k\|,$$

and the fact that $x \in \ell_M(X)$ iff $\left(\|x_k\| \right)_{k=1}^{\infty} \in \ell_M$.

In general, for $h_M(X)$, a Schauder basis isn't known. But it can be easily verified that $h_M(X)$ is separable whenever X is (see [2]). As an extension of the classical case, $\ell_M(X)$ may not be separable even if X is separable. Generally, the separability of Orlicz sequence spaces depends on whether M satisfies the Δ_2-condition.

Now, for $h_M(X)$, let we give a theorem which has the same function with a basis.

Theorem 3. For $k = 1,2,\ldots$, let $I_k : X \to h_M(X)$ and $P_k : h_M(X) \to X$ be defined by

$$I_k(u) = u \otimes e_k = \left(0,0,\ldots,0, \overset{k-th\ position}{u} ,0,\ldots \right) \text{ and } P_k(x) = x_k,$$

respectively. Then, for each $x \in h_M(X)$

$$x = \sum_{k=1}^{\infty} (I_k \circ P_k)(x),$$

that is $\left\| x - \sum_{k=1}^{n} (I_k \circ P_k)(x) \right\|_{(M)} \to 0$ as $n \to \infty$.

Proof.

$$\left\| x - \sum_{k=1}^{n} (I_k \circ P_k)(x) \right\|_{(M)} = \|(0,0,\ldots,0,x_{n+1},x_{n+2},\ldots)\|_{(M)}$$
$$= \inf\left\{ \rho > 0 : \sum_{k=n+1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) \le 1 \right\}.$$

Since $\sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) < \infty$ for every $\rho > 0$ we can find some positive integer $m = m(\rho)$ such that $\sum_{k=m+1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) \le 1$. This shows that

$$\inf\left\{ \rho > 0 : \sum_{k=n+1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) \le 1 \right\} \to 0 \text{ as } n \to \infty .$$

Lemma 4. Let M be an Orlicz function. The sets

$$\Lambda_1 = \left\{ x \in s(X) : \sum_{k=1}^{\infty} M\left(\|x_k\|\right) \le 1 \right\}$$

and

$$\Lambda_2 = \left\{ x \in s(X) : \|x\|_{(M)} \le 1 \right\}$$

are identical.

Proof. Let $x \in \Lambda_1$, this means $\sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) \le 1$ for $\rho = 1$. Hence, $\|x\|_{(M)} \le 1$, i.e., $x \in \Lambda_2$. Conversely, let $x \in \Lambda_2$, that is

$$\inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) \le 1 \right\} \le 1 .$$

This means $\sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho}\right) \le 1$ for some $\rho \le 1$. Therefore $\sum_{k=1}^{\infty} M\left(\|x_k\|\right) \le 1$ since M is non-decreasing.

4. Characterizations of Operators

Our main result is the following theorem which states the representation of operators from $h_M(X)$ into another Banach space Y.

Theorem 5. Let X, Y be Banach spaces and M, N be mutually complementary Orlicz functions. Then, $\mathrm{B}(h_M(X),Y)$ is equivalent by the mapping $T \to (T \circ I_k)$ to the Banach space

$$E_N = \left\{ A = (A_k) \in s(\mathrm{B}(X,Y)) : \|A\| = \sup_{f \in B_{Y^*}} \left\| (A_k^* f)_{k=1}^{\infty} \right\|_N < \infty \right\},$$

where each I_k is defined as in Theorem 3. By an equivalence, we mean a one to one, onto, linear isometry.

Proof. It can be easily verified that E_N is a Banach space with the norm $\|A\|$. Let $T \in \mathrm{B}(h_M(X),Y)$ and say $A_k = T \circ I_k$ for each k. This implies $\|(A_k \circ P_k)(x)\| = 0$ so that $A_k(x_k) = 0$ for each k. Since each $x \in h_M(X)$ has the representation $x = \sum_{k=1}^{\infty} (I_k \circ P_k)(x)$ we can write

$$Tx = \sum_{k=1}^{\infty} (T \circ I_k)(x_k) = \sum_{k=1}^{\infty} A_k(x_k).$$

Immediately each $A_k \in \mathrm{B}(X,Y)$ since $\|A_k\| \le \|T\| \cdot \|I_k\| = \|T\|$. Now, let us define the mapping

$$\Psi : \mathrm{B}(h_M(X),Y) \to E_N, \text{ by } \Psi(T) = A = (A_k)_{k=1}^{\infty}; \; A_k = T \circ I_k.$$

$\Psi(T) = 0$ if and only if each $T \circ I_k = 0$ and so $T = 0$ by the definition of each I_k, i.e. Ψ is one to one. Also, for an arbitrary $A \in E_N$, if we define the operator T by

$$Tx = \sum_{k=1}^{\infty} A_k(x_k)$$

on $h_M(X)$ then, by using the Young inequality, we have

$$\begin{aligned}
\left\| \sum_{k=m}^{n} A_k(x_k) \right\| &= \sup_{f \in B_{Y^*}} \left| f\left(\sum_{k=m}^{n} A_k(x_k) \right) \right| \\
&\le \sup_{f \in B_{Y^*}} \sum_{k=m}^{n} \|A_k^* f\| \cdot \|x_k\| \\
&\le \sup_{f \in B_{Y^*}} \sum_{k=m}^{n} N\left(\frac{\|A_k^* f\|}{\left\|(A_k^* f)_{k=1}^{\infty}\right\|_N} \right) + \sup_{f \in B_{Y^*}} \sum_{k=m}^{n} M\left(\left\|(A_k^* f)_{k=1}^{\infty}\right\|_N \|x_k\| \right) \\
&\le \sup_{f \in B_{Y^*}} \sum_{k=m}^{n} N\left(\frac{\|A_k^* f\|}{\left\|(A_k^* f)_{k=1}^{\infty}\right\|_N} \right) + \sum_{k=m}^{n} M\left(\frac{\|x_k\|}{1/\|A\|} \right).
\end{aligned}$$

Since $(A_k^* f)_{k=1}^{\infty} \in \ell_N(\mathrm{B}(X,Y))$ for each $f \in Y^*$ and

$$\sum_k N\left(\frac{\|A_k^* f\|}{\|(A_k^* f)_{k=1}^{\infty}\|_N} \right) \le 1$$

from the cite [8, Proposition 8.12], we have

$$\sup_{f \in B_{Y^*}} \sum_{k=m}^{n} N\left(\frac{\|A_k^* f\|}{\|(A_k^* f)_{k=1}^{\infty}\|_N} \right) \to 0 \text{ as } m,n \to \infty .$$

Also

$$\sum_{k=m}^{n} M\left(\frac{\|x_k\|}{1/\|A\|} \right) \to 0 \text{ as } m,n \to \infty$$

since $x \in h_M(X)$. This means the series $\sum A_k(x_k)$ is convergent, i.e., T is well-defined. Further, that the mapping Ψ is onto, i.e., $T \in \mathrm{B}(h_M(X),Y)$ comes from the following equalities chain

$$\begin{aligned}
\|T\| &= \sup_{x \in B_M} \|Tx\| = \sup_{x \in B_M} \left| \sum_{k=1}^{\infty} A_k(x_k) \right| \\
&= \sup_{x \in B_M} \sup_{f \in B_{Y^*}} \left| f\left(\sum_{k=1}^{\infty} A_k(x_k) \right) \right| \\
&= \sup_{f \in B_{Y^*}} \sup_{x \in B_M} \left| \sum_{k=1}^{\infty} (A_k^* f)(x_k) \right| \\
&= \sup_{f \in B_{Y^*}} \sup \left\{ \left| \sum_{k=1}^{\infty} (A_k^* f)(x_k) \right| : \sum_{k=1}^{\infty} M\left(\|x_k\| \right) \le 1 \right\}, \text{(by Lemma 4)} \\
&= \sup_{f \in B_{Y^*}} \|(A_k^* f)_{k=1}^{\infty}\|_N = \|A\|.
\end{aligned}$$

This shows, in the same time, Ψ is an isometry.

Example 6. Let $X = Y = c_0$ and M, N be mutually complementary Orlicz functions.

$$h_M(c_0) = \left\{ x \in s(c_0) : \sum_{k=1}^{\infty} M\left(\frac{\|x_k\|_\infty}{\rho} \right) < \infty, \text{for every } \rho > 0 \right\}$$

Let

$$A_k = \begin{pmatrix} 1/k^2 & 0 & 0 & 0 & \cdots \\ 1/(2k^2) & 1/(2k^2) & 0 & 0 & \cdots \\ 0 & 1/(2k^2) & 1/(2k^2) & 0 & \cdots \\ 0 & 0 & 1/(2k^2) & 1/(2k^2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, k = 1,2,\ldots \ .$$

$\mathrm{B}(c_0, c_0)$ is equivalent to infinite matrix class (c_0, c_0) since c_0 is a BK-spaces and has AK-property [9, p.218]. So, each $A \in \mathrm{B}(c_0, c_0)$ since $\|A_k\| = \sup_n \sum_i |a_{ni}^k| = k^{-2} < \infty$. Now, we assert that the sequence $A = (A_k)$ defines a continuous linear operator from $\ell_M(c_0)$ into c_0 by the virtue of Theorem 5. To show this, we shall denote that

$$\|A\| = \sup_{f \in B_{c_0^*}} \left\| (A_k^* f)_{k=1}^{\infty} \right\|_N < \infty.$$

Using the unit vector bases of c_0 and $c_0^* = \ell_1$, we can easily show that

$$A_k^* = \begin{pmatrix} 1/k^2 & 1/(2k^2) & 0 & 0 & \cdots \\ 0 & 1/(2k^2) & 1/(2k^2) & 0 & \cdots \\ 0 & 0 & 1/(2k^2) & 1/(2k^2) & \cdots \\ 0 & 0 & 0 & 1/(2k^2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, k = 1,2,\ldots \ .$$

Hence

$$A_k^* f = y_k = (y_k^n)_{n=1}^{\infty};\ y_k^n = \frac{a_n + a_{n+1}}{2k^2}$$

for each $f \in \ell_1$. Further, for each $f = (a_1, a_2, \ldots) \in B_{\ell_1}$ and for every $\rho > 0$,

$$\sum_{k=1}^{\infty} N\left(\frac{\|A_k^* f\|_1}{\rho}\right) \le \sum_{k=1}^{\infty} N\left(\frac{1}{k^2 \rho}\|f\|_1\right)$$
$$\le N\left(\frac{1}{\rho}\right)\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6} N\left(\frac{1}{\rho}\right)$$

since

$$\left\|A_k^* f\right\|_1 = \left\|y_k\right\|_1 = \sum_{n=1}^{\infty} \frac{\left|a_n + a_{n+1}\right|}{2k^2} \leq \frac{1}{k^2}\left\|f\right\|_1.$$

If we say

$$K = \inf\left\{\rho > 0 : N\left(\frac{1}{\rho}\right) \leq \frac{6}{\pi^2}\right\},$$

then $\left\|(A_k^* f)_{k=1}^{\infty}\right\|_{(N)} \leq K$ for each $f \in B_{\ell_1}$. We get

$$\sup_{f \in B_{\ell_1}} \left\|(A_k^* f)_{k=1}^{\infty}\right\|_N \leq 2K < \infty$$

Since

$$\left\|(A_k^* f)_{k=1}^{\infty}\right\|_N \leq 2\left\|(A_k^* f)_{k=1}^{\infty}\right\|_{(N)} \leq 2K.$$

Remark 7. $\ell_N(\mathrm{B}(X,Y)) \subseteq E_N$. This follows from the fact that

$$\left\|A_k^* f\right\| \leq \left\|A_k^*\right\| \cdot \left\|f\right\| = \left\|A_k\right\| \cdot \left\|f\right\|.$$

Indeed, if $(A_k) \in \ell_N(\mathrm{B}(X,Y))$ and $f \in B_{Y^*}$, we can write that

$$\sum_{k=1}^{\infty} N\left(\frac{\left\|A_k^* f\right\|}{\rho}\right) \leq \sum_{k=1}^{\infty} N\left(\frac{\left\|A_k\right\|}{\rho}\right),$$

whence $\left\|(A_k^* f)_{k=1}^{\infty}\right\|_{(N)} \leq \left\|(A_k)_{k=1}^{\infty}\right\|_{(N)}$ so that $\left\|(A_k^* f)_{k=1}^{\infty}\right\|_N \leq 2\left\|(A_k)_{k=1}^{\infty}\right\|_{(N)}$ from Lemma 2. This implies

$$\sup_{f \in B_{Y^*}} \left\|(A_k^* f)_{k=1}^{\infty}\right\|_N \leq 2\left\|(A_k)_{k=1}^{\infty}\right\|_{(N)} < \infty.$$

Example 8. The inclusion relation in Remark 7 may be strict. Let $X = \ell_1$, $Y = h_M$, and M, N be mutually complementary Orlicz functions such that $M(1) = 1$. Define $A_k : \ell_1 \to h_M$ by

$$A_k x = x_k \otimes e_k = \left(0,0,\ldots,0,\overset{k-th\ position}{x_k},0,\ldots\right)$$

for $x=(x_n)\in \ell_1$. Since h_M^* is equivalent to ℓ_N by the cite [5, p.148], for some $f\in B_{h_M^*}$

$$f(x)=\sum_n x_n y_n$$

where $y=(y_n)\in \ell_N$ and $\|y\|_{(N)}=\|f\|$. Hence, $(A_k^* f)(x)=f(A_k x)=x_k y_k$ so that

$$\|A_k^* f\|=|y_k|.$$

This implies

$$\sum_{k=1}^{\infty} N\left(\frac{\|A_k^* f\|}{\|y\|_{(N)}}\right)\le 1,$$

and so

$$\sup_{f\in B_{h_M^*}} \sum_{k=1}^{\infty} N\left(\frac{\|A_k^* f\|}{\|y\|_{(N)}}\right)\le 1.$$

This means $(A_k)\in E_N$. On the other hand, $\|A_k\|=1$ for each k. Let us show this assertion. For $x\in B_{h_M}$

$$\begin{aligned}\|A_k x\|_{(M)} &= \|x_k\otimes e_k\|_{(M)}=\inf\left\{\rho>0: M\left(\frac{|x_k|}{\rho}\right)\le 1\right\}\\ &= |x_k|, (\text{since } M(1)=1)\\ &\le \|x\|,\end{aligned}$$

that is, $\frac{\|A_k x\|_{(M)}}{\|x\|}\le 1$. Also, for $x=e_k$, $\|A_k x\|_{(M)}=1$, whence the assertion proved. Therefore, $(A_k)\notin \ell_N(\mathrm{B}(\ell_1,h_M))$ since $\sum_{k=1}^{\infty} N(1/\rho)=\infty$ for every $\rho>0$.

In the case Y is finite dimensional, Dvoretzky-Rogers theorem assert that the inclusion relation (7) is an equivalence. To show this let us recall that m_0 is the space of all scalar sequences taking on only finitely many values, that is $m_0=span\{\zeta\}$, where ζ is the set of all sequences of zeros and ones.

Theorem 9. $E_N=\ell_N(\mathrm{B}(X,Y))$ when Y is finite dimensional.

Proof. Suppose that $(A_k)\in E_N$ and $x=(x_k)\in h_M(X)$. For each $v=(v_k)\in m_0$ we have $vx=(v_k x_k)\in h_M(X)$. Indeed, for each $\rho>0$

$$\sum_{k=1}^{\infty} M\left(\frac{\|v_k x_k\|}{\rho}\right)\le \sum_{k=1}^{\infty} M\left(\frac{|v_k|\cdot\|x_k\|}{\rho}\right)\le \sum_{k=1}^{\infty} M\left(\frac{\|x_k\|}{\rho/\lambda}\right)<\infty,$$

where $\lambda=\max|v_k|$. Further,

$$\sum_k A_k(v_k x_k)$$

converges for all $v\in m_0$, since the operator $Tx=\sum_{k=1}^{\infty} A_k(x_k)$ well-defined on $h_M(X)$ by the virtue of Theorem 5. Now let (k_i) be a strictly increasing sequence of natural numbers, and define $b=(b_k)$ by

$$b_k=\begin{cases}1 & , \text{ if } k=k_i\\ 0 & , \text{ if } k\neq k_i\end{cases}.$$

Then, obviously $b\in m_0$, and so

$$\sum_k A_k(b_k x_k)=\sum_i A_{k_i}(x_{k_i})$$

is convergent. This implies that the series $\sum_k A_k(x_k)$ is subseries convergent, so it is unconditionally convergent. Hence, from Dvoretzky-Rogers theorem it is absolutely convergent since Y is finite dimensional. So, we have that

$$\sum_k \|A_k(x_k)\|<\infty$$

for each $x=(x_k)\in h_M(X)$. Now, we can find some $y_k\in B_X$ such that

$$\|A_k\|\le 2\|A_k y_k\|$$

for each k, since each $A_k\in \mathrm{B}(X,Y)$. Further, define the sequence $z=(z_k)$ such that $z=u_k y_k$ for each $(u_k)\in h_M$. Obviously $z\in h_M(X)$ and so

$$\begin{aligned}\sum_k \|A_k\| \cdot |u_k| &\le 2\sum_k |u_k| \cdot \|A_k y_k\| \\ &= 2\sum_k \|A_k(u_k y_k)\| \\ &= 2\sum_k \|A_k z_k\| < \infty.\end{aligned}$$

This shows that the real sequence $\left(\|A_k\|\right)_{k=1}^{\infty} \in h_M^{\alpha}$. But, from the cite [8], α-dual of h_M is equivalent to ℓ_N. Hence,

$$\sum_{k=1}^{\infty} N\left(\frac{\|A_k\|}{\rho}\right) < \infty$$

for some $\rho > 0$. This means $A = (A_k) \in \ell_N(\mathrm{B}(X,Y))$.

Corollary 10. Let M, N be mutually complementary Orlicz functions. Then $[h_M(X)]^* = \ell_N(X^*)$.

This is a direct consequence of Theorem 5 and Theorem 9 for $Y = \mathrm{X}$, the scalar field of $h_M(X)$ and X.

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