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ON CR-SUBMANIFOLDS HAVING HOLOMORPHIC VECTOR FIELDS ON THEM

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ABSTRACT

This paper studies the CR-submanifolds of a Kaehlerian manifold which have holomorphic vector fields on them. It is shown that a CR-submanifold having holomorphic vector fields on it is a CR-product.

1. INTRODUCTION

The notion of a CR-(Cauchy-Riemann) submanifold of a Kaehlerian manifold was firstly introduced by A. Bejancu [1]. Afterward a lot of authors concerned with the subject. In this study, it is considered the notion of holomorphic vector field (given in [3]) for CR-submanifolds having vector fields on them. We may discuss the integrability conditions of distrubutions and the necessary conditions of the leaves of the distributions to be totally geodesic.

2. BASIC CONCEPTS

In this section we give the fundamental concepts concerning with the study

Let $\overline{M}$ be a Riemann manifold and M be a submanifold of $\overline{M}$. The Riemannian metric g on $\overline{M}$ induces a Riemannian metric on M. Let TM and $TM^{\perp}$ denote tangent and normal bundle, respectively, and $\overline{\nabla}$ and ∇ be the Levi-Civita connections on $\overline{M}$ and M, respectively, Then for X, Y $\in$ $\Gamma(TM)$ we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

where $\Gamma(TM)$ is the module of differentiable sections defined on the bundle TM and h is the second fundamental form of M. The equation (2.1) is called as the Gauss formula. V being an element of $\Gamma(TM^{\perp})$ the Weingarten formula is given by

$$\overline{\nabla}_X V = - A_V X + \nabla^{\perp}_X V \tag{2.2}$$

where A_V is the fundamental tensor of Weingarten with respect to the normal section V, and $\nabla^{\perp}$ is the normal connection on M. It is well known that

$$g(h(X, Y), V) = g(A_V X, Y) \tag{2.3}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(TM^{\perp})$.

Let $\overline{M}$ be a Riemannian manifold. Let g and J be Riemannian metric and a tensor field of type (1, 1) on $\overline{M}$ and M, respectively. Then $\overline{M}$ is called a Kaehlerian manifold if the following conditions are satisfied.

1) $J^2 = -I$

2) $g(JX, JY) = g(X, Y)$, $X.Y \in \Gamma(T\overline{M})$

3) $(\overline{\nabla}_X J)Y = 0$

where I denotes the identity transformation of $\Gamma(T\overline{M})$ [2]. The vector field X on $\overline{M}$ is called as holomorphic vector field if $L_X J = 0$ where LX is the Lie derivative with respect to X [3].

A vector field X is holomorphic if and only if

$$J\overline{\nabla}_V X = \overline{\nabla}_{JV} X \tag{2.4}$$

where X and V belong to $\Gamma(T\overline{M})$ [3].

Let $\overline{M}$ be a Kaehlerian manifold and M be a real submanifold of $\overline{M}$. It is said that M is a CR-submanifold of $\overline{M}$ if there are distributions D and $D^{\perp}$ satisfying the conditions [1].

1) $T_M(p) = D_p \oplus D_P^{\perp}$

2) $J(D) = D$, $J(D^{\perp}) \subset TM^{\perp}$

we denote p and q the complex dimension of the distribution D and the real dimension of the distribution $D^{\perp}$, respectively, then for $q = 0$ (resp. $p = 0$) a CR-submanifold becomes a complex submanifold (resp. totally real submanifold). M is called as anti-holomorphic submanifold if $\dim D_X^{\perp} = \dim T_M^{\perp}(x)$. For CR-submanifolds it can be written

$$JX = \phi X + \omega X \tag{2.5}$$

where ϕX and ωX are the tangential part and the normal part of JX, respectively [1]. ν being the orthogonal complement of $JD^{\perp}$ i.e. $TM^{\perp} = JD^{\perp} \oplus \nu$, for each $V \in \Gamma(TM^{\perp})$ we can with

$$JV = BV + CV \tag{2.6}$$

where $BV \in \Gamma(D^{\perp})$ and $CV \in \Gamma(\nu)$. It is well known that the distribution D is integrable if and only if $[X, Y] \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$ [4].

Theorem 2.1. Let $\overline{M}$ be a Kaehlerian manifold and M be a CR-submanifold of $\overline{M}$. Then the distribution D is integrable if and only if the second fundamental form of M satisfies [2], for $X, Y \in \Gamma(D)$

$$h(X, JY) = h(JX, Y). \tag{2.7}$$

3. CR-SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD HAVING HOLOMORPHIC VECTOR FIELDS ON THEM

First we give the lemma

Lemma 3.1. Let $\overline{M}$ be a Kaehlerian manifold and M be a CR-submanifold of $\overline{M}$ such that there are some holomorphic vector fields defined on M. Then, for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, we have

$$g(h(X, Y) + h(JX, JY), JZ) = 0\ . \tag{3.1}$$

Proof. By using (2.1) and (2.4) we get

$$\nabla_{JY}X + h(JY, X) = J\nabla_Y X + Jh(Y, X), \tag{3.2}$$

thus, we have

$$g(\nabla_{JY}X, Z) = g(J\nabla_Y X + Jh(Y, X), Z)$$
$$= -g\ (H(Y, X), JZ)$$

for any $Z \in \Gamma(D^{\perp})$. Hence we obtain

$$g(h(Y, X), JZ) = g(J\overline{\nabla}_{JY}JX, Z)$$
$$= -g(\overline{\nabla}_{JY}JX, JZ)$$

or

$$g(h(Y, X), JZ) = -g(h(JY, JX), JZ)$$

this completes the proof of the lemma.

Theorem 3.1 Let $\overline{M}$ be a Kaehlerian manifold and M be a CR-submanifold of $\overline{M}$ having holomorphic vector fields on it. Then D is integrable and each leaf of D is totally geodesic on M.

Proof. Since M has holomorphic vector field on it we have

$$J\overline{\nabla}_X Y = \overline{\nabla}_{JX} Y$$

for any $X, Y \in \Gamma(TM)$. Considering

$$\overline{\nabla}_{JX} Y = \overline{\nabla}_X JY$$

we may write

$$\nabla_{JX} Y = h(JX, Y) = \nabla_X JY + h(X, JY).$$

Hence we get

$$h(JX, Y) = h(X, JY)$$

threfore, from theorem (2.1), D is integrable. Now we are going to show that leaves of D are totally geodesic on M. For $Z \in \Gamma(D^{\perp})$ we have

$$g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, Z)$$

or

$$g(\nabla_X Y, Z) = g([X, Y], Z) + g(\nabla_Y X, Z),$$

since D is integrable we get

$$g(\nabla_X Y, Z) = g(\nabla_Y X, Z)$$
$$= g(\overline{\nabla}_Y JX, JZ)$$

or

$$g(\nabla_X Y, Z) = -g(JX, \overline{\nabla}_Y JZ).$$

From (2.2) we obtain

$$g(\nabla_X Y, Z) = g(JX, A_{JZ} Y)$$
$$= g((A_{JZ} JX, Y)$$
$$= -g(\overline{\nabla}_{JX} JZ, Y)$$
$$= -g(\overline{\nabla}_{JX} JZ, Y)$$
$$= -g(J \overline{\nabla}_X JZ, Y)$$
$$= g(\overline{\nabla}_X Z, Y)$$
$$= -g(Z, \overline{\nabla}_X Y)$$

or

$$g(\nabla_X Y, Z) = -g(\nabla_X Y, Z)$$

from the last equation, we see that

$$g(\nabla_X Y, Z) = 0.$$

This implies that $\nabla_X Y \in \Gamma(D)$, which proves the assertion.

From Theorem 3.1 we obtain the following result:

Corollary 3.1. Let $\overline{M}$ be a Kaehlerian manifold and M be a CR-submanifold of $\overline{M}$ having holomorphic vector fields on it. Then each leaf of D is totally geodesic on $\overline{M}$ if and only if we have

$$(L_V g)(X, Y) = 0$$

for any $X, Y \in \Gamma(D)$ and $V \in \Gamma(\nu)$.

Proof. From Theorem 3.1 we have

$$g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, Z) = 0$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Since $D^\perp$ is anti-invariant under J, there exist a nonzero vector field $W \in \Gamma(D^\perp)$ such that $\xi = JW$ for $\xi \in \Gamma(JD^\perp)$. Thus we obtain

$$\begin{aligned} g(\overline{\nabla}_X Y, \xi) &= g(\overline{\nabla}_X Y, JW) \\ &= -g(\overline{\nabla}_X JY, W) \\ &= 0. \end{aligned}$$

On the other hand, since the Levi-Civita connection of $\overline{M}$ is given

$$\begin{aligned} 2g(\overline{\nabla}_X Y, V) = {} & X(g(Y, V)) + Y((V, X)) - V(g(X, Y)) + \\ & g([X,Y], V) + g([V, X], Y) - g([Y, V], X) \end{aligned}$$

we have

$$2g(\overline{\nabla}_X Y, V) = -V(g(X,Y)) + g([V, X], Y) + g([V, Y], X)$$

for any $X, Y \in \Gamma(D)$ and $V \in \Gamma(\nu)$. Hence

$$2g(\overline{\nabla}_X Y, V) = -(L_V g)(X, Y) .$$

This proves our assertion.

Theorem 3.2. Let $\overline{M}$ be a Kaehlerian manifold and M be a CR-submanifold of $\overline{M}$ having holomorphic vector fields on it. Then each maximal integral manifold of $D^\perp$ is totally geodesic on M.

Proof. $Z, W \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$ we have

$$\begin{aligned} g(\nabla_W Z, X) &= g(\overline{\nabla}_W Z, X) \\ &= g\,(J\overline{\nabla}_W Z, JX) \end{aligned}$$

or

$$g(\nabla_W Z, X) = g(\overline{\nabla}_W JZ, JX)$$
$$= Wg(JZ, JX) - g(JZ, \overline{\nabla}_W JX)$$

since $JZ \in \Gamma(TM^{\perp})$ and $JX \in \Gamma(D)$ we obtain

$$g(\nabla_W Z, X) = -g(JZ, \overline{\nabla}_W JX)$$
$$= -g(JZ, J\overline{\nabla}_W X)$$
$$= -g(JZ, \overline{\nabla}_{JW} X)$$
$$= -JWg(JZ, X) + g(\overline{\nabla}_{JW} JZ, X)$$
$$= g(\overline{\nabla}_{JW} JZ, X)$$
$$= -g(\nabla_W Z, X)$$

or

$$2g(\nabla_W Z, X) = 0.$$

Because of the last equation we have $\nabla_W Z \in \Gamma(D^{\perp})$ which implies that each maximal integral manifold of $D^{\perp}$ is totally geodesic on M.

Combining Theorem 3.1 with Theorem 3.2 we have the following Corollary.

Corollary 3.2. Let $\overline{M}$ be a Kaehlerian manifold and M be a CR-submanifold of $\overline{M}$ having holomorphic vector fields on it. Then M is a CR-product.

Proof. Let M_1 and M_2 be the maximal integral manifold of D and $D^{\perp}$ on CR-submanifold, respectively. The locally Riemann product $M_1 \times M_2$ is called as a CR-product. $M_1 \times M_2$ is a locally Riemann product if and only if both distributions D and $D^{\perp}$ are integrable and the maximal integral manifolds of them are totally geodesic in M[2]. By virtue of Theorem (2.1) and Theorem (3.2) and Lemma 3.3 in [5], $M_1 \times M_2$ is a CR-product.

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