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THE CONJUGATE OF A HYPERSURFACE

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ABSTRACT

In this study, the idea of the conjugate of a surface in E^3 given by TH. Hasanis and D. Koutroufiotis [3] has been generalized for a hypersurface in E^{n+1}. A necessary and sufficient condition for having the conjugate of a hypersurface has been given. Gauss and mean curvatures of the conjugate hypersurface have also been calculated.

1. INTRODUCTION

Let M be a smooth immersed regular hypersurface in E^{n+1}, which is connected and oriented. Let us choose $O \in E^{n+1}$ as an origin. We denote by x the position vector of a point in M, and set $|x| = r$ for the corresponding distance function. Let N be the unit normal vector field of M. The support function f of M with respect to O is defined as $f = -\langle x, N\rangle$, which is also differentiable, where $\langle\ ,\ \rangle$ is the inner product on E^{n+1}. Let $(u^1, \ldots, u^n)$ be a local coordinate system on M. We denote the components of the first, second and third fundamental forms, respectively, by $g_{ij} = \langle x_i, x_j\rangle$, $b_{ij} = -\langle x_i, N_j\rangle$ and $n_{ij} = \langle N_i, N_j\rangle$, where $x_i = \dfrac{\partial x}{\partial u^i}$ and $N_i = \dfrac{\partial N}{\partial u^i}$.

Let $\overline{\nabla}$ be the standard connection of E^{n+1}, ∇ be the induced connection on M. The equations of Gauss and Weingarten are, respectively,

$$\overline{\nabla}_X Y = \nabla_X Y + (AX, Y)\, N, \tag{1.1}$$

and

$$\overline{\nabla}_X N = -\ AX \tag{1.2}$$

where X and Y are vector fields tangent to M and A is the Weingarten mapping of M. The eingenvalues of A are the principal curvatures

$k_1, k_2, \ldots k_n$. The Gauss curvature is $K = k_1 k_2 \ldots k_n$ and the mean curvatures is $H = \frac{1}{n} \sum_{i=1}^{n} k_i$.

Suppose now that there exist a point O with the property that it lies on no tangent hyperplane of M. If we choose such a point as origin, the corresponding support function clearly never vanishes. So, either $f > 0$ or $f < 0$. We can always choose an orientation of M which makes $f > 0$. Thus, M is obviously star-shaped.

We decompose the position vector x of a point of M into two parts a component normal to M, and a component tangent to M such that

$$x = x_T - fN. \tag{1.3}$$

Let X be a tangent vector of M. Since $\overline{\nabla}_X x = X$,

$$X = \overline{\nabla}_X x = \overline{\nabla}_X\left(x_T - fN\right) = \overline{\nabla}_X x_T - (Xf)N - f\overline{\nabla}_X N$$

or

$$X = \nabla_X x_T + \langle AX, x_T \rangle N - (Xf)N + f\ AX.$$

Taking the tangential component of this equation, we obtain

$$\nabla_X x_T = (I - fA)X, \tag{1.4}$$

where I is the identity transformation, and taking the normal component we obtain

$$\langle AX, x_T \rangle N = (Xf)N$$

or

$$\langle X, Ax_T \rangle = \langle X, \text{grad } f \rangle.$$

So that

$$Ax_T = \text{grad } f. \tag{1.5}$$

Furthermore, since

$$\begin{aligned} X(r^2) &= X(\langle x, x\rangle) \\ &= 2\langle \overline{\nabla}_X x, x\rangle \\ &= 2\langle X, x_T\rangle, \end{aligned}$$

or

$$\begin{aligned} X(r^2) &= 2rX(r) \\ &= 2r\langle X, \text{grad } r\rangle, \end{aligned}$$

then

$$\text{grad } r = \frac{x_T}{r}, \tag{1.6}$$

2. THE CHARACTERISTIC MAPPING OF A HYPERSURFACE

Let M be oriented hypersurface and S^n be the unit hypersphere centered at O. We define the smooth mapping $\zeta : M \rightarrow S^n$ by

$$\zeta(x) = \frac{x + 2fN}{r}$$

Further, we define the mapping $\eta : M \rightarrow S^n$ by

$$\eta(x) = e = \frac{x}{r} ,$$

that is, η is a diffeomorphism of M onto the open subset $A = \eta(M)$ of S^n. Then we can define the characteristic mapping $\tau : A \rightarrow S^n$ of M, where $\tau : \zeta \circ \eta^{-1}$ by. Obviously, the position vector e of a point in A with respect to O can be written as

$$\tau(e) = e + \frac{2\, f\, N}{r} . \tag{2.1}$$

Let $(u^1, \ldots, u^n)$ be a local coordinate system of A, so we write $e_i = \dfrac{\partial e}{\partial u^i}$ and $\tau_i = \dfrac{\partial \tau}{\partial u^i}$. From (2.1)

$$1 - \langle \tau(e), e\rangle = \frac{2f^2}{r^2} . \tag{2.2}$$

Then, τ can have no fixed points. Instead of $\tau(e)$, we write simply τ and using $e = \frac{x}{r}$, after a brief calculation we obtain

$$\langle \tau, e_i\rangle = \frac{2f^2}{r^2} \frac{\partial}{\partial u^i} (\log r), \quad 1 \leq i \leq n . \tag{2.3}$$

From (2.2) and (2.3), we find the first-order system of differential equations

$$\frac{\partial}{\partial u^i}(\log r) = \frac{\langle \tau, e_i \rangle}{1 - \langle \tau, e \rangle}, \quad 1 \le i \le n . \tag{2.4}$$

The integrability conditions for this system, can be written as

$$\frac{\partial}{\partial u^i}\left[\frac{\langle \tau, e_j \rangle}{1 - \langle \tau, e \rangle}\right] = \frac{\partial}{\partial u^j}\left[\frac{\langle \tau, e_i \rangle}{1 - \langle \tau, e \rangle}\right], \ 1 \le i , j \le n ,$$

or

$$\langle \tau_j, e_i \rangle - \langle \tau_i, e_j \rangle = \frac{\langle \tau, e_j \rangle \langle \tau_i, e \rangle - \langle \tau, e_i \rangle \langle \tau_j, e \rangle}{1 - \langle \tau, e \rangle} . \tag{2.5}$$

The length of the position vector r of M satisfies the differential equations system (2.4). If a given mapping $\tau : A \to S^n$ without fixed points is the characteristic mapping of a hypersurface, then the corresponding hypersurface M is given by its position vector x = re.

3. THE CONJUGATE OF A HYPERSURFACE

Let S^n be unit hypersphere centered at O and e be the position vector of S^n. The mapping $\alpha : S^n \to S^n$, $\alpha(e) = -e$, is called as an antipodal mapping. If a given the characteristic mapping τ of a hypersurface M, we set $\bar{\tau} = \alpha \circ \tau$.

Definition 3.1. Let τ be the characteristic mapping of a hypersurface M in E^{n+1}. If $\bar{\tau}$ also the characteristic mapping of some hypersurface $\bar{M}$, then $\bar{M}$ is called the conjugate hypersurface of M.

If $\bar{\tau}$ is the characteristic mapping of an $\bar{M}$, then $\bar{\tau}$ has no fixed points.

Theorem 3.2. The hypersurface M has the conjugate $\bar{M}$ if and only if grad r $\neq$ 0 and the vector field grad r, grad f on M are linear depended at every point.

Proof. Suppose M has the conjugate $\bar{M}$. Then $\bar{\tau}$ has no fixed points, that is, $\tau(e) \neq -e$ for every e in the domain of τ. This means that x is never perpendicular to M, and since grad $r = \sum_{i=1}^{n} r_i \frac{\partial}{\partial u^i}$, $r_i = \frac{\partial r}{\partial u^i} = \frac{\langle x, x_i \rangle}{r}$, grad $r \neq 0$. Considering the integrability condition (2.5) for τ and $\bar{\tau}$, we obtain

$$\langle \tau_i, e_j \rangle = \langle \tau_j, e_i \rangle \ . \tag{3.1}$$

From (3.1), we compute

$$\langle \bar{\tau}_i, e_j \rangle - \langle \bar{\tau}_j, e_i \rangle = \frac{4f}{r^3}\left(r_j f_i - f_j r_i\right) = 0 \ ,$$

or

$$f_i r_j = f_j r_i \ .$$

Thus, the vector fields grad r, grad f are linear depended.

Conversely grad $r \neq 0$ and the vector fields grad r, grad f are linear depended. Since grad $r \neq 0$ the mapping $\bar{\tau} = \alpha \circ \tau$ has no fixed points. Since the grad r and grad f are linear depended, the equality (3.1) holds. Hence, the $\bar{\tau}$ satisfies the integrability condition (2.5), that is M has the conjugate $\bar{M}$.

Theorem 3.2 holds for a hypersurface M. From (1.5) and (1.6)

$$Ax_T = \text{grad } f = c \text{ grad } r = \frac{c}{r} x_T \ , \quad c \neq 0 \ , \quad c \in \mathbb{R},$$

this means the vector x_T is the eigen vector of A. Thus, M has conjugate hypersurface if and only if the tangential component x_T of the position vector x of M is the eigen vector of A. Setting $X = x_T$ in (1.4), we obtain

$$\nabla_{x_T} x_T = (1 - fk_1)x_T,$$

where k_1 is the principal curvature the corresponding to x_T.

Since the position vector of M can be written as $x = re$, we write $\bar{x} = \bar{r}e$, where $\bar{x}$ is the position vector of $\bar{M}$. Moreover $\frac{x}{r} = \frac{\bar{x}}{\bar{r}}$ and $\bar{\tau}(e) = -\tau(e)$. So,

$$\frac{x}{r} + \frac{2\,f\,N}{r} = -\,\frac{x}{r} - \frac{2\,\bar{f}\,\bar{N}}{\bar{r}} \ .$$

This relation tells us that $\bar{N}$ is the hyperplane spanned by x and N. We compute $\langle \bar{N}, N \rangle = 0$, hence $\bar{N}$ is parallel to x_T. For the position vector of $\bar{M}$, we write

$$\bar{x} = \frac{\bar{r}}{r}\, x = \frac{\bar{r}}{r}\left(x_T - f\, N\right),$$

or

$$\bar{x} = \bar{x}_T - \bar{f}\,\bar{N}\,.$$

From this we obtain $\bar{f} = -\langle \bar{x}, \bar{N}\rangle = -\frac{\bar{r}}{r}\langle x_T, \bar{N}\rangle$. Since $\bar{N}$ is parallel to x_T, we choose $\bar{N} = \frac{-x_T}{|x_T|}$, which makes $\bar{f}$ positive and

$$\bar{f} = \frac{\bar{r}}{r}\,|x_T|\,.$$

Theorem 3.3. The natural mapping from M to $\overline{M}$ preserves principal directions. Moreover, the corresponding principal curvatures at corresponding points are related by

$$\bar{k}_1 = \frac{\bar{f}\, r^2}{f\, \bar{r}^2}\, k_1\ ,\quad \bar{k}_i = \frac{1 - f k_i}{\bar{f}}\ ,\ 2 \le i \le n\ ,$$

where k_1 is the principal curvature in the direction x_T.

Proof. Let $(u^1, u^2, ..., u^n)$ be the local coordinate system in the neighbourhood of a point of M which is not an umbilic. Let the parameter curves of M be the curvature lines. Since, M has the conjugate $\overline{M}$, the curves u^j = sbt. $2 \le j \le n$, are the integral curves of the vectorfield x_T. Thus $g_{ij} = b_{ij} = 0$ and $k_i = \frac{b_{ii}}{g_{ii}}$. Moreover, $r = r(u^1)$ and $f = f(u^1)$ because x_T is parallel x_1. We can write the position vector x of M with respect to the basis $\{x_1, ..., x_n, N\}$ of E^{n+1},

$$x = \sum_{i=1}^{n} c_i\, x_i + c_{n+1}\, N.$$

We compute the coefficients, $c_i = \frac{\langle x, x_i\rangle}{g_{ii}} = \frac{r r_i}{g_{ii}}$ and $c_{n+1} = -f$. Since $r_i = 0$, $i \neq 1$, we obtain

$$x = \frac{r r_1}{g_{11}}\, x_1 - f\, N\,. \tag{3.2}$$

From (1.3) and (3.2)

$$x_T = \frac{r r_1}{g_{11}}\, x_1\ ,\ r r_1 = \sqrt{g_{11}}\ |x_T|\,.$$

Since $|x_T|^2 = r^2 - f^2$, the g_{11} depends on u^1 only. We differentiate (3.2) with respect to u^i,

$$x_i = \frac{\text{r}r_1}{g_{11}} x_{1i} - f N_i \ , \ 2 \le i \le n \ .$$

Using Rodrigues formula $N_i = -k_i x_i$, we get

$$\frac{(1 - f k_i)}{|x_T|} x_i = \frac{1}{\sqrt{g_{11}}} x_{1i} \ .$$

If we product both sides of the last equation with x_i then

$$\frac{(1 - fk_i)}{|x_T|} = \frac{1}{2\sqrt{g_{11}}} \frac{\partial}{\partial u^i} (\log g_{1i}), \ 2 \le i \le n \ .$$

Since $\overline{N} = \frac{-x_T}{|x_T|}$, we can take as $\overline{N} = \frac{-x_1}{\sqrt{g_{11}}}$. Set $h = \frac{\overline{r}}{r}$, then $\overline{x} = hx$ and

$$\overline{x}_i = h_i x + hx_i,$$

where $h_1 = \frac{-h\sqrt{g_{11}}}{|x_T|}$ and $h_i = 0, 2 \le i \le n$. Thus, $\overline{g}_{11} = \frac{h^2 f^2}{|x_T|} g_{11}, \overline{g}_{ij} = 0 \ , i \ne j$

$\overline{g}_{ii} = h^2 g_{ii}$, $2 \le i \le n$. Similarly $\overline{b}_{11} = \frac{hf}{|x_T|} b_{11}$, $i \ne j$, $\overline{b}_{ii} = \frac{h}{2\sqrt{g_{11}}} \frac{\partial g_{ii}}{\partial u^1}$, $2 \le i \le n$. Therefore, the parameter curves of $\overline{M}$ are the lines of curvature, so that the natural mapping preserves principal directions.

For the principal curvatures of M, we obtain

$$\overline{k}_1 = \frac{\overline{b}_{11}}{\overline{g}_{11}} = \frac{\overline{f}\, r^2}{f\, \overline{r}^2} k_1 \ ,$$

and

$$\overline{k}_i = \frac{\overline{b}_{ii}}{\overline{g}_{ii}} = \frac{1}{2h\sqrt{g_{11}}} \frac{\partial}{\partial u^1} (\log g_{ii}) = \frac{1 - fk_i}{\overline{f}} \ , \ 2 \le i \le n \ .$$

This completes the proof.

Corollary. The Gauss curvature of $\overline{M}$ is

$$\overline{K} = \frac{k_1}{h^{-2} f \overline{f}^{n-2}} \left[1 - f \sum_{i=2}^{n} k_i + f^2 \sum_{\substack{i=2 \\ i<j}}^{n} k_i k_j - f^3 \sum_{\substack{i=2 \\ i<j<l}}^{n} k_i k_j k_l + \dots \right.$$

$$\left. \dots - f^{n-2} \sum_{i=2}^{n} k_1 \dots \hat{k}_i \dots k_n \right] + \frac{f^{n-2}}{\overline{h}^2 \overline{f}^{n-2}} K,$$

in which K is the Gauss curvature of M are $\hat{k}_i$ is meant dropping i-th curvature function k_i of M.

Corollary 3.5. The mean curvature of $\overline{M}$ is

$$\overline{H} = \frac{(n-1)f + r^2 k_1}{nf\bar{f}} - \frac{f}{\bar{f}} H ,$$

where H is the mean curvature of M.

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