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Some properties of the first eigenvalue of the $p(x)$-laplacian on riemannian manifolds

R. A. Mashiyev, G. Alisoy and S. Ogras

Abstract

The main result of the present paper establishes a stability property of the first eigenvalue of the associated problem which deals with the $p(x)$-Laplacian on Riemannian manifolds with Dirichlet boundary condition.

Key Words: Variable exponent Lebesgue and Sobolev spaces; first eigenvalue; Riemannian manifolds; $p(x)$-Laplacian.

1. Introduction

Over the last decades the variable exponent Lebesgue spaces $L^{p(x)}$ and the corresponding Sobolev space $W^{1,p(x)}$ have been a subject of active research stimulated by development of the studies of problems in elasticity, fluid dynamics, calculus of variations, and differential equations with $p(x)$-growth (see [2], [3], [12]). We refer the reader to [5], [7], [8] for fundamental properties of these spaces.

The $p(x)$-Laplacian equations related to eigenvalue problems have been studied in [6], [9], [10], [11].

Let $G \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with a smooth boundary. For measurable function $p(x)$ we denote the variable exponent Lebesgue space by

$$L^{p(x)}(G) = \left\{ u \text{ measurable real functions} : \int_G |u(x)|^{p(x)}\, dx < \infty \right\},$$

which is equipped with the norm, the so-called Luxemburg norm (see [5], [7])

$$|u|_{p(x)} := |u|_{L^{p(x)}(G)} = \inf \left\{ \delta > 0 : \int_G \left| \frac{u(x)}{\delta} \right|^{p(x)} dx \leq 1 \right\},$$

where

$$1 < ess \inf_{x\in G} p(x) := p^- \leq p(x) \leq ess \sup_{x \in G} p(x) := p^+ < \infty,$$

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where $\left(L^{p(x)}(G), |\cdot|_{L^{p(x)}}\right)$ becomes a Banach space, which is known as a variable exponent Lebesgue space.

Define the variable exponent Sobolev space $W^{1,p(x)}(G)$ by

$$W^{1,p(x)}(G) = \{u \in L^{p(x)}(G); \quad |\nabla u| \in L^{p(x)}(G)\},$$

and equip with the norm

$$\|u\|_{1,p(x)} := \|u\|_{W^{1,p(x)}(G)} = |u|_{p(x)} + |\nabla u|_{p(x)} \ , \forall u \in W^{1,p(x)}(G).$$

The space $W_0^{1,p(x)}(G)$ is denoted by the closure of $C_0^\infty(G)$ in $W^{1,p(x)}(G)$ which is equipped with the norm for $u \in W_0^{1,p(x)}(G)$

$$\|u\|_{1,p(x)} = |\nabla u|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(G).$$

If $p^- > 1$, then the spaces $L^{p(x)}(G), W^{1,p(x)}(G)$ and $W_0^{1,p(x)}(G)$ are separable and reflexive Banach spaces (see [5], [7]).

Proposition 1. ([5], [7]). *Denote*

$$\varrho_{\ p(x)}(u) = \int_G |u(x)|^{\ p(x)}\, dx,$$

and

$$\varrho_{\ 1,p(x)}(u) := \varrho_{\ p(x)}(\nabla u) = \int_G |\nabla u(x)|^{\ p(x)}\, dx, \forall u, \nabla u \in L^{p(x)}(G),$$

then we have

$$\min\left\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right\} \le \varrho_{\ p(x)}(u) \le \max\left\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right\},$$

$$\min\left\{|\nabla u|_{p(x)}^{p^-}, |\nabla u|_{p(x)}^{p^+}\right\} \le \varrho_{\ 1,p(x)}(u) \le \max\left\{|\nabla u|_{p(x)}^{p^-}, |\nabla u|_{p(x)}^{p^+}\right\}.$$

Let M be a compact Riemannian manifold with $\dim M = m$, and $\triangle_{p(x)}$ is nonhomegenous $p(x)$-Laplacian acting on functions on M, where $\triangle_{p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$, and $1 < p(x) < \infty$.Let M^* be a compact submanifold of Ω, and B_ε the tubular neinghborhood of M^* of radius $\varepsilon > 0$; that is,

$$B_\varepsilon = \{x \in M : d(x, M^*) < \varepsilon\},$$

where $d(.,.)$ is the distance function on M induced by the Riemannian metric. Donete by $\triangle_{p(x),\varepsilon}$ the restriction of $\triangle_{p(x)}$ to those functions on M vanishing identically in B_ε. Set

$$\Omega_\varepsilon = M \backslash B_\varepsilon \text{ and } \partial\Omega_\varepsilon = \partial B_\varepsilon.$$

We consider the following Dirichlet problem

$$\begin{cases} \triangle_{p(x),\varepsilon} u + \lambda(\varepsilon) |u|^{p(x)-2} u = 0 & x \in B_\varepsilon \\ u = 0 & x \in \partial B_\varepsilon. \end{cases}$$

The first eigenvalue $\lambda_{1,p(x)}(\Omega_\varepsilon, \phi)$ of the $p(x)$-Laplacian is defined as the least number λ for which the Dirichlet problem has a nontrivial solution $u \in W_0^{1,p(x)}(\Omega_\varepsilon)$. It can be characterized by

$$\lambda_{1,p(.)}(\Omega_\varepsilon, \phi) = \inf_{u \in W_0^{1,p(x)}(\Omega_\varepsilon) \setminus \{0\}} \frac{\int_{\Omega_\varepsilon} |\nabla u|^{p(x)} d\upsilon_\phi}{\int_{\Omega_\varepsilon} |u|^{p(x)} d\upsilon_\phi},$$

where u runs over $W_0^{1,p(x)}(\Omega_\varepsilon)$ and $d\upsilon_\phi$ denotes the volume element of M. It turns out $\lambda_{1,p(x)}(\Omega_\varepsilon, \phi) > 0$. When $B_\varepsilon = \varnothing$, that is, $\Omega_\varepsilon = M$; u runs over $W^{1,p(x)}(\Omega_\varepsilon)$ and $\int_M |u|^{p(x)-2} u d\upsilon_\phi = 0$. We can easily see that $\lambda_{1,p(x)}(M) = 0$. The corresponding eigenfunctions are constant functions. In the case of $p = 2$, many people have studied the asymptotic exponsion of the eigenvalues $\lambda_{k,2}(\varepsilon)$ $(k = 1, 2, ...)$ for the $2-$Laplacian of a manifold $M \backslash B_\varepsilon$ with the Dirichlet condition on the tubular neighborhood B_ε. Chavel and Feldman in [4] showed that $\lambda_{k,2}(\varepsilon) \to 0$ as $\varepsilon \to 0$ tends to zero under the condition codim $(M^*) \geq 2$. Eigenvalue problems for quasilinear operators of p-Laplace type have received considerable attention in the last years (see [1]).

In the present paper, we establish conditions ensuring a stability property of the first eigenvalue of the associated problem which deals with the $p(x)$-Laplacian. As far as we are concerned, this is the first paper that discusses this subject. We prove that the first eigenvalues of the $p(x)$-Laplacian on Riemannian manifolds Ω_ε converges to zero as ε tends to zero. In precisely we show $\lambda_{1,p(x)}(\Omega_\varepsilon, \phi) \to \lambda_{1,p(x)}(M, \phi) = 0$ as $\varepsilon \to 0$.

Moreover, the investigation of the Riemannian manifolds in the variable exponent Lebesgue space $L^{p(x)}$ is firstly dealt with in our study.

2. Main results and proofs

Lemma 1. *Suppose that the codimension $m - k \geq p^+$. Given any $f \in W^{1,p(x)}(M)$, there exists a function $f_\varepsilon \in W_0^{1,p(x)}(\Omega_\varepsilon)$ such that*

$$\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_{1,p(x)} = \lim_{\varepsilon \to 0} |f_\varepsilon - f|_{p(x)} + \lim_{\varepsilon \to 0} |\nabla(f_\varepsilon - f)|_{p(x)} = 0$$

for sufficiently small $\varepsilon > 0$.

Proof. By Proposition 1, we can write

$$\varrho_{\ 1,p(x)}(f_\varepsilon - f) = \varrho_{\ p(x)}(f_\varepsilon - f) + \varrho_{\ p(x)} \nabla (f_\varepsilon - f).$$

Define a function $\omega_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\omega_\varepsilon(r) = \begin{cases} 1, \varepsilon \le r; \\ \left(\frac{r}{\varepsilon}\right)^\varepsilon, \exp\left(-\varepsilon^{-2}\right) \le r \le \varepsilon; \\ \left(\frac{\exp(-\varepsilon^{-2})}{\varepsilon}\right)^\varepsilon \left(\frac{2r}{\exp(-\varepsilon^{-2})}\right), 2^{-1}\exp\left(-\varepsilon^{-2}\right) \le r \le \exp\left(-\varepsilon^{-2}\right) \\ 0,\ r \le \exp\left(-\varepsilon^{-2}\right). \end{cases}$$

Then we have

$$\left|\omega_\varepsilon^{'}(r)\right|^{p(r)} = \begin{cases} 0, \varepsilon \le r \\ \left(\frac{r}{\varepsilon}\right)^{\varepsilon p(r) - p(r)}, \exp\left(-\varepsilon^{-2}\right) \le r \le \varepsilon \\ \left(\frac{\exp(-\varepsilon^{-2})}{\varepsilon}\right)^{\varepsilon p(r)} \left(\frac{2}{\exp(-\varepsilon^{-2})}\right)^{p(r)}, 2^{-1}\exp\left(-\varepsilon^{-2}\right) \le r \le \exp\left(-\varepsilon^{-2}\right) \\ 0, r \le \exp\left(-\varepsilon^{-2}\right). \end{cases}$$

Define $f_\varepsilon = \omega_\varepsilon f \in W_0^{1,p(x)}(\Omega_\varepsilon)$ and $r(x) = d(M^*, x)$. Then we have

$$\varrho_{\ p(x)}(f_\varepsilon - f) = \int\limits_M |(f_\varepsilon - f)|^{p(x)}\, d\upsilon_\phi,$$

and

$$\lim_{\varepsilon\to 0} \int\limits_\varepsilon^r |f_\varepsilon - f|^{p(x)}\, d\upsilon_\phi = \lim_{\varepsilon\to 0} \int\limits_\varepsilon^r |(\omega_\varepsilon - 1) f|^{p(x)}\, d\upsilon_\phi \to 0,$$

and

$$\lim_{\varepsilon\to 0} \int\limits_{\exp(-\varepsilon^{-2})}^{\varepsilon} \left|\left(\left(\frac{r}{\varepsilon}\right)^\varepsilon - 1\right).f\right|^{p(x)} d\upsilon_\phi \to 0, (\lim_{\varepsilon\to 0} \varepsilon^\varepsilon \to 1),$$

$$\lim_{\varepsilon\to 0} \int\limits_{2^{-1}\exp(-\varepsilon^{-2})}^{\exp\left(-\varepsilon^{-2}\right)} \left|\left(\left(\frac{\exp(-\varepsilon^{-2})}{\varepsilon}\right)^\varepsilon \left(\frac{2r}{\exp(-\varepsilon^{-2})}\right) - 1\right).f\right|^{p(x)} d\upsilon_\phi \to 0,$$

and

$$\lim_{\varepsilon\to 0} \int\limits_r^{\exp\left(-\varepsilon^{-2}\right)} |f_\varepsilon - f|^{p(.)}\, d\upsilon_\phi \to 0;$$

so we have

$$\lim_{\varepsilon\to 0} \int\limits_M |f_\varepsilon - f|^{\ p(x)}\, d\upsilon_\phi = 0.$$

Hence, by Proposition 1 we can write $|f_\varepsilon - f|_{p(x)} = 0$.

Let's consider

$$\varrho_{\,p(x)}\nabla\left(f_\varepsilon - f\right) = \int\limits_M |\nabla(f_\varepsilon - f)|^{p(x)}\,d\upsilon_\phi = \int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |\nabla(f_\varepsilon - f)|^{p(x)}\,d\upsilon_\phi + \int\limits_{B_{2^{-1}\exp(-\varepsilon^{-2})}} |\nabla f|^{p(x)}\,d\upsilon_\phi,$$

where $T_{\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} = B_\varepsilon \backslash B_{2^{-1}\exp(-\varepsilon^{-2})}$. Since $f \in W^{1,p(x)}(M)$,

$$\int\limits_{B_{2^{-1}\exp(-\varepsilon^{-2})}} |\nabla f|^{p(x)}\,d\upsilon_\phi \to 0 \text{ as } 2^{-1}\exp(-\varepsilon^{-2}) \to 0,$$

and

$$\int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |\nabla(f_\varepsilon - f)|^{p(x)}\,d\upsilon_\phi \le 2^{p^+-1}\Big(\int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |\nabla f_\varepsilon|^{p(x)}\,d\upsilon_\phi + \int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |\nabla f|^{p(x)}\,d\upsilon_\phi\Big).$$

The first term on the ring hand side tends to 0 as $\varepsilon \to 0$ because of $f \in W^{1,p(x)}(M)$. By the definition of f_ε, we have

$$\int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |\nabla f_\varepsilon|^{p(x)}\,d\upsilon_\phi \le \int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |(\nabla\omega_\varepsilon)f + \omega_\varepsilon(\nabla f)|^{p(x)}\,d\upsilon_\phi$$

$$\le 2^{p^+-1}\Big(\int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |f|^{p(x)}\,|\nabla\omega_\varepsilon|^{p(x)}\,d\upsilon_\phi + \int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |\nabla f|^{p(x)}\,|\omega_\varepsilon|^{p(x)}\,d\upsilon_\phi\Big).$$

Since $f \in W^{1,p(x)}(M)$, the term $\int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |\nabla f|^{p(x)}\,|\omega_\varepsilon|^{p(x)}\,d\upsilon_\phi \to 0$ as $\varepsilon \to 0$.

Next we shall show

$$\int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |f|^{p(x)}\,|\nabla\omega_\varepsilon|^{p(x)}\,d\upsilon_\phi \to 0 \text{ as } \varepsilon \to 0.$$

From $\dim M^* = k$ and $m - k \ge p^+$, that is, the codimension of M^* and M is grater than p^+ or egual to p^+. We consider the fact

$$p^- \le \inf_{x\in(\varepsilon,\exp(-\varepsilon^{-2}))} p(x) \text{ and } \sup_{x\in(\varepsilon,\exp(-\varepsilon^{-2}))} p(x) \le p^+,$$

with which we can estimate

$$\int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |f|^{p(x)}\,|\nabla\omega_\varepsilon|^{p(x)}\,d\upsilon_\phi \le C^{p^+} \int\limits_{T\varepsilon,2^{-1}\exp(-\varepsilon^{-2})} |\nabla\omega_\varepsilon|^{p(x)}\,d\upsilon_\phi$$

$$\le 2^{p^+-1}C^{p^+}\Big(C_1 \int\limits_{\exp(-\varepsilon^{-2})}^{\varepsilon} \left(\frac{r}{\varepsilon}\right)^{\varepsilon p(r)-p(r)} r^{m-k-1}dr + C_2 \int\limits_{2^{-1}\exp(-\varepsilon^{-2})}^{\exp(-\varepsilon^{-2})} \left(\frac{\exp(-\varepsilon^{-2})}{\varepsilon}\right)^{\varepsilon p(r)}$$

$$\times \left(\frac{2}{\exp(-\varepsilon^{-2})}\right)^{p(r)} r^{m-k-1}dr) \leq C_1\left(p^+\right) \int\limits_{\exp(-\varepsilon^{-2})}^{\varepsilon} \left(\frac{r}{\varepsilon}\right)^{\varepsilon p^- - p^+} r^{m-k-1}dr$$

$$+\frac{C_2\left(p^+\right)}{\varepsilon^{\varepsilon p^+}} \exp\left(-\left(\varepsilon p^+ - p^+\right)/\varepsilon^2\right) \int\limits_{2^{-1}\exp(-\varepsilon^{-2})}^{\exp\left(-\varepsilon^{-2}\right)} r^{m-k-1}dr \;= \frac{C_1\left(p^+\right)}{\varepsilon^{\varepsilon p^- - p^+}} \left[\frac{r^{\varepsilon p^- - p^+ + m-k}}{\varepsilon p^- - p^+ + m - k}\right]_{\exp(-\varepsilon^{-2})}^{\varepsilon}$$

$$+\frac{C_2\left(p^+\right)}{\varepsilon^{\varepsilon p^+}} \exp\left(-\left(\varepsilon p^+ - p^+\right)/\varepsilon^2\right) \left[\frac{r^{m-k}}{m-k}\right]_{2^{-1}\exp(-\varepsilon^{-2})}^{\exp\left(-\varepsilon^{-2}\right)}$$

$$\leq \frac{C_1\left(p^+\right)}{p^-\varepsilon^{\varepsilon p^- - p^+ + 1}} \left\{\varepsilon^{\varepsilon p^- - p^+ + m-k} - \exp(-(\varepsilon p^- - p^+ + m - k)/\varepsilon^2)\right\}$$

$$+\frac{C_2\left(p^+, m, k\right)}{\varepsilon^{\varepsilon p^+}} \left\{\exp\left(-\varepsilon^{-2}(\varepsilon p^+ - p^+ + m - k\right)\right\}$$

$$= \frac{C_1\left(p^+\right)}{p^-}\varepsilon^{m-k-1} - \frac{C_1\left(p^+\right)\varepsilon^{p^+-1}}{p^-\varepsilon^{\varepsilon p^-}} \exp(-\left(\varepsilon p^- - p^+ + m - k\right)/\varepsilon^2)$$

$$+\frac{C_2\left(p^+, m, k\right)}{\varepsilon^{\varepsilon p^+}} \exp(-\left(\varepsilon p^+ - p^- + m - k\right)/\varepsilon^2).$$

The right hand side tends to 0 by using

$$\lim_{\varepsilon\to 0} \varepsilon^{m-k-1} = 0 \text{ (because of } \; m-k \geq p^+ - 1 > 0), \;\; \lim_{\varepsilon\to 0} \varepsilon^{p^+-1} \exp(-\left(\varepsilon p^- - p^+ + m - k\right)/\varepsilon^2) = 0$$

$$\text{(because of } \; \lim_{\varepsilon\to 0} \varepsilon^{\varepsilon} = 1), \lim_{\varepsilon\to 0} \frac{\exp(-\left(\varepsilon p^- - p^+ + m - k\right)/\varepsilon^2)}{\varepsilon^{\varepsilon p^-}} = 0.$$

This completes the proof of Lemma 1. □

Theorem 2. *Let (M, ϕ) be a compact Riemannian manifold with $dimM = m$, and M^* a closed submanifold in M with $dimM^* = k$. Suppose that the codimension $m - k \geq p^+$ of M^* in M is grater than or equal to p^+. Let $\lambda_{1,p(x)}\left(\Omega_\varepsilon, \phi\right)$ be the first eigenvalue of the $p(x)$-Laplacian on Ω_ε. We have*

$$\lim_{\varepsilon\to 0} \lambda_{1,p(x)}\left(\Omega_\varepsilon, \phi\right) = \lambda_{1,p(x)}\left(M, \phi\right) = 0.$$

The corresponding eigenfunctions ϕ_ε in Ω_ε converge to a constant function ϕ_1 in $L^{p(x)}(M)$.

Proof. Take the eigenfunction f for $\lambda_{1,p(x)}\left(M, \phi\right)$ such that $\int\limits_M |f|^{p(x)}\, d\upsilon_\phi = 1$. Then

$$\lambda_{1,p(x)}\left(M\right) = \int\limits_M |\nabla f|^{p(x)}\, d\upsilon_\phi.$$

From Lemma 1, there exists $f_\varepsilon \in W_0^{1,p(x)}(\Omega_\varepsilon)$ such that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |f_\varepsilon - f|^{p(x)}\, d\upsilon_\phi \to 0 \text{ and } \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla(f_\varepsilon - f)|^{p(x)}\, d\upsilon_\phi \to 0.$$

Thus

$$\lambda_{1,p(x)}(\Omega_\varepsilon,\phi) \le \frac{\int_{\Omega_\varepsilon} |\nabla f_\varepsilon|^{p(x)}\, d\upsilon_\phi}{\int_{\Omega_\varepsilon} |f_\varepsilon|^{p(x)}\, d\upsilon_\phi} \to \frac{\int_M |\nabla f|^{p(x)}\, d\upsilon_\phi}{\int_M |f|^{p(x)}\, d\upsilon_\phi} = \lambda_{1,p(x)}(M,\phi)\,(\varepsilon\to 0). \tag{1}$$

Now let ϕ_ε be the first eigenvalue for the $p(x)$-Laplacian on Ω_ε. By formula(1), ϕ_ε is uniformly bounded. ϕ_ε has a strongly convergence limit $\phi_1 \in L^{p(x)}(M)$ in $L^{p(x)}(M)$. By formula (1), and the compactness of ϕ_ε, we have $\int_M |\phi_1|^{p(x)}\, d\upsilon_\phi = 1$ and

$$\lambda_{1,p(.)}(M,\phi) \le \int_M |\nabla\phi_1|^{p(x)}\, d\upsilon_\phi \le \lim_{\varepsilon\to 0}\int_M |\nabla\phi_\varepsilon|^{p(x)}\, d\upsilon_\phi \le \lambda_{1,p(x)}(M,\phi).$$

This shows that $\lambda_{1,p(x)}(\Omega_\varepsilon,\phi) \to \lambda_{1,p(x)}(M,\phi)$ as $\varepsilon \to 0$, and the limit function $\phi_1 \in L^{p(x)}(M)$ is the first eigenfunction for $p(x)$-Laplacian on M.

This completes the proof of Theorem. □

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R. A. MASHIYEV
Department of Mathematics,
Dicle University, 21280 Diyarbakır-TURKEY
e-mail:mrabil@dicle.edu.tr

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Gulizar ALİSOY
Department of Mathematical Sciences,
İnonu University, 44280 Malatya-TURKEY
e-mail:galisoy@inonu.edu.tr

Sezai OGRAS
Department of Mathematics,
Dicle University, 21280 Diyarbakır-TURKEY
e-mail:sezaio@dicle.edu.tr