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On convergence of double sequences of closed sets

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Abstract. In this paper, we extend the definitions of various kinds of convergence from ordinary (single) sequences to double sequences of closed sets. We examine the relationship among them. Also, we introduce monotone double sequences of sets and analyze the limit of monotone double sequences of sets.

Key words. Set-valued function, double sequence of sets, Kuratowski convergence, Hausdorff convergence, Wijsman convergence, Fisher convergence.

Kapalı kümelerin çift dizilerinin yakınsaklığı üzerine

Özet. Bu çalışmada, kapalı kümelerin tek dizileri için verilen yakınsaklık çeşitlerini çift dizilere genişlettik. Bu yakınsaklık çeşitleri arasındaki ilişkileri inceledik. Bir de monoton çift küme dizilerini tanımlayarak, bu dizilerin limitlerini araştırdık.

Anahtar kelimeler. Küme değerli fonksiyonlar, çift küme dizileri, Kuratowski yakınsaklık, Hausdorff yakınsaklık, Wijsman yakınsaklık, Fisher yakınsaklık.

1 Introduction

The concept of convergence for double sequences was initially introduced by Pringsheim [25] in the 1900s. Since then, this concept has been studied by many authors, [1, 8, 12, 21, 22, 26, 29, 30, 36].

Set-valued functions are an important mathematical notion and play a crucial role in several practical areas. Continuity properties of a set-valued mapping can be defined on the basis of convergence of sequence of sets [2, 10, 15, 16, 18, 20, 27, 35]. There are different convergence notions for sequence of sets, which have significance for certain applications. The best known of them are Kuratowski convergence [17], Hausdorff convergence [13, 14], Wijsman convergence [33, 34] and Fisher convergence [11]. Concerning these types of convergence, the reader could consult the book of G. Beer [4] and the survey paper of Baronti and Papini [3]. See also [5, 7, 19, 31, 32].

The purpose of this paper is to extend basic results known in the literature from ordinary (single) sequences of sets to double ones.

The plan of the paper is as follows: In Section 2, we give some fundamental definitions and the basic notations for the different types of convergence of sets. In Section 3, we give the related results on Kuratowski convergence for double sequences of closed sets. In Section 4, we emphasize on the other types of convergence for double sequences of closed sets. Also, the relations among various types of convergence are investigated. In the final section, we examine monotone double sequences of sets.

2 Definitions and notation

A double sequence $x = (x_{jk})$ is said to be convergent to l in the Pringsheim [25] sense (briefly as P-convergent) if for given $\varepsilon > 0$ there exists an integer n_0 such that $|x_{jk} - l| < \varepsilon$ whenever $j, k > n_0$. We write this as

$$\lim_{j,k\to\infty} x_{jk} = l,$$

where j and k tend to infinity independent of each other. We denote by $\mathcal{C}_p$, the space of P-convergent double sequences. Throughout this paper limit of a double sequence means limit in the Pringsheim sense.

A double sequence $x = (x_{jk})$ is said to be Cauchy double sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{pq} - x_{jk}| < \varepsilon$ for all $p \geq j \geq N$, $q \geq k \geq N$.

A double sequence x is bounded if

$$\|x\| = sup_{j,k}|x_{jk}| < \infty.$$

Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded.

The idea of convergence in the Pringsheim sense can be extended to a double sequence of points of a metric space. We say that a double sequence $x = (x_{jk})$ of points of a metric space (X, d) P-convergence to a point $l \in X$ if

$$\lim_{j,k\to\infty} d(x_{jk}, l) = 0.$$

Patterson [23] gave the definition of subsequence and the Pringsheim limit point of a double sequence.

A number $L \in X$ is said to be a Pringsheim limit point of a double sequence (x_{jk}) if there exist two strictly increasing sequences (j_i) and (k_i) such that

$$\lim_{i\to\infty} x_{j_i k_i} = L.$$

The set of all Pringsheim limit points of a double sequence (x_{jk}) will be denoted by P_x.

Patterson [24] gave the definition of the Pringsheim limit inferior and limit superior of double sequences of real numbers.

Definition 2.1 [24] *Let $x = (x_{kl})$ be a double sequence of real numbers and for each n, let $\alpha_n = \sup_n\{x_{kl} : k, l \geq n\}$. The Pringsheim limit superior of x is defined as follows:*

(i) if $\alpha_n = +\infty$ for each n, then $\limsup_{k,l\to\infty} x_{kl} := +\infty$;

(ii) if $\alpha_n < +\infty$ for some n, then $\limsup_{k,l\to\infty} x_{kl} := \inf_n\{\alpha_n\}$.

Similarly, let $\beta_n = \inf_n\{x_{kl} : k, l \geq n\}$ then the Pringsheim limit inferior of $x = (x_{kl})$ is defined as follows:

(i) if $\beta_n = -\infty$ for each n, then $\liminf_{k,l\to\infty} x_{kl} := -\infty$;

(ii) if $\beta_n > -\infty$ for some n, then $\liminf_{k,l\to\infty} x_{kl} := \sup_n\{\beta_n\}$.

Let (X, d) be a metric space and $A \subset X$, $x \in X$. Then the distance from a point x to a subset A of X is given by

$$d(x, A) := \inf_{a\in A} d(x, a),$$

where we set $d(x, \emptyset) := \infty$. As long as A is closed, having $d(x, A) = 0$ is equivalent to having $x \in A$.

For each closed subset A of X, the distance function $x \to d(., A)$ is Lipschitz continuous, i.e., for each $x, y \in X$,

$$|d(x, A) - d(y, A)| \leq d(x, y). \tag{2.1}$$

For sets A and B in X, the excess of A beyond B is defined by

$$\delta(A,B)=\sup_{x\in A} d(x,B) \quad \text{if} \quad A\neq\emptyset\ ; \quad \delta(\emptyset,B)=0.$$

The Hausdorff distance between two sets A and B, denoted by $h(A,B)$, is defined as follows:

$$h(A,B)=\max\left(\delta(A,B),\delta(B,A)\right)\ \ (\leq +\infty) \tag{2.2}$$

unless both A and B are empty in which case $h(A,B)=0$. Note that if only one of the two sets is empty then $h(A,B)=\infty$.

Equivalently, the Hausdorff distance between two nonempty sets A and B in X can be expressed by

$$h(A,B) \quad = \quad \sup_{x\in X}|d(x,A)-d(x,B)|. \tag{2.3}$$

The open ball with center x and radius $\varepsilon>0$ in X is denoted by $B(x,\varepsilon)$, i.e.,

$$B(x,\varepsilon)=\{y\in X \mid d(x,y)<\varepsilon\}.$$

For any set A and $\varepsilon>0$, we denote the open ε-enlargement of A by A^{ε}, i.e.,

$$A^{\varepsilon}=\{x\in X: d(x,A)<\varepsilon\}=\bigcup_{x\in A} B(x,\varepsilon).$$

Note that A^{ε} is convex if A is convex. Also,

$$A=\bigcap_{\varepsilon>0} A^{\varepsilon} \quad \text{and} \quad \overline{A^{\varepsilon}}=\{x\in X: d(x,A)\leq\varepsilon\}.$$

For any $B\subset X$, we have $\delta(B,A)=\inf\{\varepsilon>0: B\subset A^{\varepsilon}\}$. By $\Omega(x)$, we denote the set of neighborhoods of x.

Let $(E,\|\cdot\|)$ be a real normed vector space. For $u,v\in E$, we donete by $[u,v]$ the closed segment joining u and v, i.e.,

$$[u,v]=\{\lambda u+(l-\lambda)v:\lambda\in[0,1]\}.$$

Also, $e_n=(0,\ldots,0,1,0,\ldots)$, where 1 is at n^{th} place.

Let us recall definition of Kuratowski, Hausdorff, Wijsman and Fisher convergence of sets.

We use the following notation:

$$\begin{aligned}
\mathcal{N} &:= \{N\subseteq\mathbb{N}:\mathbb{N}\backslash N \ \text{ finite}\}\\
&:= \{\text{subsequences of } \mathbb{N} \text{ containing all n beyond some } n_0\},\\
\mathcal{N}^{\#} &:= \{N\subseteq\mathbb{N}: N \text{ infinite}\}=\{\text{all subsequences of } \mathbb{N}\}.
\end{aligned}$$

Definition 2.2 [17] *Let (X,d) be a metric space. For a sequence (A_n) of subsets of X; the upper limit is the set*

$$\begin{aligned}\limsup_{n\to\infty} A_n &:= \left\{x \mid \forall\ V\in\Omega(x),\ \exists N\in\mathcal{N}^{\#},\ \forall n\in N: A_n\cap V\neq\emptyset\right\}\\ &:= \left\{x \mid \exists N\in\mathcal{N}^{\#},\ \forall n\in N,\ \exists x_n\in A_n: \lim_{n\in N} x_n = x\right\}\end{aligned}$$

while the lower limit is the set

$$\begin{aligned}\liminf_{n\to\infty} A_n &:= \left\{x \mid \forall\ V\in\Omega(x),\ \exists N\in\mathcal{N},\ \forall n\in N: A_n\cap V\neq\emptyset\right\}\\ &:= \left\{x \mid \exists N\in\mathcal{N},\ \forall n\in N,\ \exists x_n\in A_n: \lim_{n\in N} x_n = x\right\}.\end{aligned}$$

The limit of the sequence of sets exists if the outer and inner limit sets are equal, that is,

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n.$$

For some properties of upper and inner limits we refer to [4,5,19,27,28,31,32].

Definition 2.3 [14] *A sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of X is said to be Hausdorff convergent to a closed subset A of X if $\lim_{n\to\infty} h(A_n,A)=0$, in which case we write $H-\lim_{n\to\infty} A_n = A$. (Note that Hausdorff convergence must be defined for closed sets since otherwise limit sets are not well-defined).*

Definition 2.4 [34] *Let (X,d) be a metric space. For any non-empty closed subsets A, $A_n\subset X$, we say that the sequence (A_n) is Wijsman convergent to A if*

$$\lim_{n\to\infty} d(x,A_n) = d(x,A)$$

for each $x\in X$. In this case we write $W-\lim_{n\to\infty} A_n = A$.

Definition 2.5 [11] *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a metric space X. (A_n) converges to A according to Fisher if the following conditions hold:*

(i) For any $\varepsilon>0$ there exists n_ε such that $\delta(A_n,A)<\varepsilon$ for $n>n_\varepsilon$,

(ii) for any $\varepsilon>0$ and $x\in A$, there exists $n_{\varepsilon,x}$ such that $d(x,A_n)<\varepsilon$ for $n>n_{\varepsilon,x}$.

In this case, we write $F-\lim_{n\to\infty} A_n = A$.

We always have the implication $H\Rightarrow F\Rightarrow W\Rightarrow K$. The opposite implication holds if there is a compact set K which contains A and every A_n (see [3, Proposition 1, Proposition 9]).

3 Kuratowski convergence for double sequence of sets

In this section, we introduce Kuratowski convergence for double sequence of sets. We give the definition of upper and lower limit of double sequence of sets. We get some equivalent conditions to the definition.

For operational reasons in handling statements about sequences, it will be convenient to work with the following collections of subsets of $\mathbb{N}^2$. We use the following notation:

$$\begin{aligned}\mathcal{N}_2 &:= \{M \subseteq \mathbb{N}^2 \mid \text{there exists } \ n \in \mathbb{N} \ \text{ such that } \ (k,l) \in M \ \text{ for all } \ k,l \geq n\} \\ &:= \{M \subseteq \mathbb{N}^2 \mid \exists n \in \mathbb{N} : (k,l) \in M, \ \forall k,l \geq n\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}_2^{\#} &:= \{M \subseteq \mathbb{N}^2 \mid \text{for all } \ n \in \mathbb{N} \ \text{ there exist } \ k,l \geq n \ \text{ such that } \ (k,l) \in M\} \\ &:= \{M \subseteq \mathbb{N}^2 \mid \forall n \in \mathbb{N}, \ \exists k,l \geq n : (k,l) \in M\}.\end{aligned}$$

Definition 3.1 *Let $(A_{kl})_{k,l\in\mathbb{N}}$ be a double sequence of subsets of a metric space X. We say that the subset*

$$\begin{aligned}\limsup_{k,l\to\infty} A_{kl} &:= \left\{x \in X \mid \forall\varepsilon > 0, \ \exists N \in \mathcal{N}_2^{\#}, \ \forall (k,l) \in N : A_{kl} \cap B(x,\varepsilon) \neq \emptyset\right\} \\ &:= \left\{x \in X \mid \forall\varepsilon > 0, \ \forall n \in \mathbb{N}, \ \exists k,l \geq n : B(x,\varepsilon) \cap A_{kl} \neq \emptyset\right\}\end{aligned}$$

is the upper limit of the double sequence (A_{kl}) and that the subset

$$\begin{aligned}\liminf_{k,l\to\infty} A_{kl} &:= \left\{x \in X \mid \forall\varepsilon > 0, \ \exists N \in \mathcal{N}_2, \ \forall (k,l) \in N : A_{kl} \cap B(x,\varepsilon) \neq \emptyset\right\} \\ &:= \left\{x \in X \mid \forall\varepsilon > 0, \ \exists n \in \mathbb{N} : B(x,\varepsilon) \cap A_{kl} \neq \emptyset, \ \forall k,l \geq n\right\}\end{aligned}$$

is its lower limit. Moreover, if there exists a set $A \subseteq X$ such that

$$A = \liminf_{k,l\to\infty} A_{kl} = \limsup_{k,l\to\infty} A_{kl},$$

then we write $\lim_{k,l\to\infty} A_{kl} = A$ (or $K_2 - \lim_{k,l\to\infty} A_{kl} = A$), and we say that the double sequence (A_{kl}) converges to A in the sense of Kuratowski.

Moreover, it's clear from the inclusion $\mathcal{N}_2 \subset \mathcal{N}_2^{\#}$ that

$$\liminf_{k,l\to\infty} A_{kl} \subseteq \limsup_{k,l\to\infty} A_{kl}$$

so that in fact, $\lim_{k,l\to\infty} A_{kl} = A$ if and only if

$$\limsup_{k,l\to\infty} A_{kl} \subseteq A \subseteq \liminf_{k,l\to\infty} A_{kl}.$$

Remark 3.1 $\lim_{k,l\to\infty} A_{kl} = A$ *if and only if the following conditions are satisfied:*

(i) For every $x \in A$ *and for every* $\varepsilon > 0$ *there exists* $n \in \mathbb{N}$ *such that* $B(x,\varepsilon) \cap A_{kl} \neq \emptyset$ *for every* $k,l \geq n$*;*

(ii) for every $x \in X \setminus A$ *there exist* $\varepsilon > 0$ *and* $n \in \mathbb{N}$ *such that* $B(x,\varepsilon) \cap A_{kl} = \emptyset$ *for every* $k,l \geq n$*.*

Example 3.1 *Define, in* $\mathbb{R}$*,* (A_{kl}) *by*

$$A_{kl} := \begin{cases} [0,l] & , \quad k=1, \\ [0,k] & , \quad l=1, \\ [0,1] & , \quad k \cdot l \text{ is even}, \ k>1, \ l>1, \\ [-1,0] & , \quad k \cdot l \text{ is odd}, \ k>1, \ l>1, \end{cases}$$

$\liminf_{k,l\to\infty} A_{kl} = \{0\}$*, and* $\limsup_{k,l\to\infty} A_{kl} = [-1,1]$.

Example 3.2 *Define, in* $\mathbb{R}$*,* (A_{kl}) *by*

$$A_{kl} := \left[\frac{(-1)^k}{k}, 2 + \frac{(-1)^l}{l}\right].$$

Then we have $\lim A_{kl} = [0,2] := A$*, whereas each column and row of the double sequence* (A_{kl}) *does not converge to* A*.*

Lower and upper limits of double sequences of sets can be described alternatively by the following formulas.

Proposition 3.2 *Let* $(A_{kl})_{k,l\in\mathbb{N}}$ *be a double sequence of subsets of a metric space* X*. Then,*

$$\liminf_{k,l\to\infty} A_{kl} = \bigcap_{N\in\mathcal{N}_2^{\#}} cl \bigcup_{(k,l)\in N} A_{kl} \quad \text{and} \quad \limsup_{k,l\to\infty} A_{kl} = \bigcap_{N\in\mathcal{N}_2} cl \bigcup_{(k,l)\in N} A_{kl}.$$

By Proposition 3.2, the sets $\liminf_{k,l\to\infty} A_{kl}$ and $\limsup_{k,l\to\infty} A_{kl}$ are closed in X. Moreover, by definition of $\mathcal{N}_2$, we have that

$$\limsup_{k,l\to\infty} A_{kl} = \bigcap_{n\in\mathbb{N}} \mathrm{cl} \bigcup_{k,l\geq n} A_{kl}.$$

Proposition 3.3 *Let* $(A_{kl})_{k,l\in\mathbb{N}}$ *be a double sequence of closed subsets of a metric space* X*. Then,*

(i) $\limsup_{k,l\to\infty} A_{kl} := \{x \in X : \liminf_{k,l\to\infty} d(x, A_{kl}) = 0\}$,

(ii) $\liminf_{k,l\to\infty} A_{kl} := \{x \in X : \lim_{k,l\to\infty} d(x, A_{kl}) = 0\}$.

The proofs can be carried out in the same way in the case of a single sequence (see [4, Proposition 5.2.2], [10, Proposition 3A.1]).

For a double sequence (A_{kl}) of nonempty sets in X upper and lower limit sets can be described equivalently in terms of the double sequences (y_{kl}) that can be formed by selecting an $y_{kl} \in A_{kl}$ for each $(k,l) \in \mathbb{N}^2$.

Proposition 3.4 *If $(A_{kl})_{k,l\in\mathbb{N}}$ is a double sequence of sets in a metric space X, then*

$$\liminf_{k,l\to\infty} A_{kl} = \Big\{x \mid \text{ there exists a double sequence } (y_{kl}), \; y_{kl} \in A_{kl} \text{ for any } k,l \in \mathbb{N}, \text{ with } \lim_{k,l\to\infty} y_{kl} = x\Big\}.$$

Proof. Sufficiency is obvious. For necessity, let $x \in \liminf_{k,l\to\infty} A_{kl}$ be arbitrary. We have for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$A_{kl} \cap B(x,\varepsilon) \neq \emptyset$$

for every $k,l \geq n_0$. Let us take $\varepsilon = \frac{1}{i}$, $i = 1,2,3,\ldots$. Then there exists $n_1 \in \mathbb{N}$ such that

$$A_{kl} \cap B(x,1) \neq \emptyset$$

for every $k,l \geq n_1$. By the same argument, there exists $n_2 \in \mathbb{N}$ such that

$$A_{kl} \cap B(\left(x, \frac{1}{2}\right) \neq \emptyset$$

for every $k,l \geq n_2$. Continuing in this way, there exists $n_i \in \mathbb{N}$ such that

$$A_{kl} \cap B\left(x, \frac{1}{i}\right) \neq \emptyset$$

for every $k,l \geq n_i$. Let us form $n_1 < n_2 < \cdots < n_i < \cdots$ and define the sequence

$$y_{kl} \in A_{kl} \cap B\left(x, \frac{1}{i}\right) \quad (k,l) \in M_i \setminus M_{i+1}, \quad i = 1,2,...,$$

where

$$M_i = \{(k,l) : k,l \geq n_i\}.$$

$y_{kl} \in A_{kl}$ can be chosen arbitrarily for $k,l < n_1$. Then, we get $\lim_{k,l\to\infty} y_{kl} = x$. ■

Proposition 3.5 *If $(A_{kl})_{k,l\in\mathbb{N}}$ is a double sequence of sets in a metric space X, then*

$$\limsup_{k,l\to\infty} A_{kl} = \Big\{x \mid \text{ there exist increasing two sequences } k_i, l_i, \; y_{k_i l_i} \in A_{k_i l_i} \text{ for any } i \in \mathbb{N}, \text{ with } \lim_{i\to\infty} y_{k_i l_i} = x\Big\}. \tag{3.1}$$

Proof. Let $x \in \limsup_{k,l\to\infty} A_{kl}$ be arbitrary. We have for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there exist $k, l \geq n$ such that

$$A_{kl} \cap B(x, \varepsilon) \neq \emptyset.$$

For $\varepsilon = 1$ and $n_1 = 1$ there exist $k_1, l_1 \geq n_1$ such that

$$A_{k_1 l_1} \cap B(x, 1) \neq \emptyset.$$

For $\varepsilon = \frac{1}{2}$ and $n_2 = \max\{k_1, l_1\} + 1$ there exist $k_2, l_2 \geq n_2$ such that

$$A_{k_2 l_2} \cap B\left(x, \frac{1}{2}\right) \neq \emptyset.$$

For $\varepsilon = \frac{1}{3}$ and $n_3 = \max\{k_2, l_2\} + 1$ there exist $k_3, l_3 \geq n_3$ such that

$$A_{k_3 l_3} \cap B\left(x, \frac{1}{3}\right) \neq \emptyset.$$

Continuing in this way, for $\varepsilon = \frac{1}{i}$ $(i = 1, 2, 3, \dots)$ and $n_i = \max\{k_{i-1}, l_{i-1}\} + 1$, there exist $k_i, l_i \geq n_i$ such that

$$A_{k_i l_i} \cap B(x, \frac{1}{i}) \neq \emptyset.$$

Hence, we can construct the sequences (k_i) and (l_i) such that

$$y_{k_i l_i} \in A_{k_i l_i} \cap B\left(x, \frac{1}{i}\right)$$

exist, i.e., $y_{k_i l_i} \in A_{k_i l_i}$ and $d(y_{k_i l_i}, x) < \frac{1}{i}$. This means that $\lim_{i\to\infty} y_{k_i l_i} = x$. Therefore x belongs to the set in the right-hand side of equality (3.1).

On the contrary, assume that x belongs to the right-hand side set of equality (3.1). Then, there exist two subsequences (k_i), (l_i) of positive integers such that $y_{k_i l_i} \in A_{k_i l_i}$ for any $i \in \mathbb{N}$ and $\lim_{i\to\infty} y_{k_i l_i} = x$. In this case, for every $\varepsilon > 0$ there exists n_0 such that $d(y_{k_i l_i}, x) < \varepsilon$ for $i > n_0$, i.e., $y_{k_i l_i} \in B(x, \varepsilon)$. Since the sequences (k_i) and (l_i) are increasing for all $n \in \mathbb{N}$, there exist $k_i, l_i \geq \max\{n, n_0\}$ such that $y_{k_i l_i} \in A_{k_i l_i} \cap B(x, \varepsilon)$, i.e.,

$$A_{k_i l_i} \cap B(x, \varepsilon) \neq \emptyset.$$

Hence $x \in \limsup_{k,l\to\infty} A_{kl}$. ∎

By Proposition 3.4 and Proposition 3.5, note that $\liminf_{k,l\to\infty} A_{kl}$ is the set of limits of double sequences (y_{kl}) with $y_{kl} \in A_{kl}$ for any $(k, l) \in \mathbb{N}^2$ and $\limsup_{k,l\to\infty} A_{kl}$ is the set of Pringsheim limit points of double sequences $y_{kl} \in A_{kl}$ for any $(k, l) \in \mathbb{N}^2$.

Corollary 3.6 *Let X be a normed linear space and (A_{kl}) be a sequence of convex subsets of X. Then $\liminf_{k,l\to\infty} A_{kl}$ is convex and so, when it exists, is $\lim_{k,l\to\infty} A_{kl}$.*

Proof. Let $\liminf_{k,l\to\infty} A_{kl} = A$. If x_1 and x_2 belong to A, by Proposition 3.4, for all $k, l \in \mathbb{N}$ we can find the points y_{kl}^1 and y_{kl}^2 in A_{kl} such that $\lim_{k,l\to\infty} y_{kl}^1 = x_1$ and $\lim_{k,l\to\infty} y_{kl}^2 = x_2$. Then for arbitrary $\lambda \in [0,1]$ let us define

$$y_{kl}^{\lambda} := (1-\lambda)y_{kl}^1 + \lambda y_{kl}^2 \quad \text{and} \quad x_\lambda := (1-\lambda)x_1 + \lambda x_2.$$

Then, $\lim_{k,l\to\infty} y_{kl}^{\lambda} = x_\lambda$. By Proposition 3.4, we obtain $x_\lambda \in A$. This means that A is convex. ■

Proposition 3.7 [3, Proposition 10] *Let X be a finite-dimensional normed linear space and (A_n) be a sequence of closed convex subsets of X. If $\lim_{n\to\infty} A_n = A \neq \emptyset$ with A compact. Then, $\bigcup_{n=1}^{\infty} A_n$ is bounded.*

Now, we give an example which shows that Proposition 3.7 is not valid for double sequences.

Example 3.3 *Define (A_{kl}) by*

$$A_{kl} := \begin{cases} [-k,k] & , \quad l = 1, \\ [2,3] & , \quad \textit{otherwise}. \end{cases}$$

Then (A_{kl}) is a double sequence of closed convex subsets of $\mathbb{R}$ and $\lim_{k,l\to\infty} A_{kl} = [2,3]$. However, $\bigcup_{k,l=1}^{\infty} A_{kl} = \mathbb{R}$ is not bounded.

4 Other types of convergence for double sequence of closed sets

In this section, we introduce three kinds of convergence for double sequence of sets. We get the relations among types of convergence.

Definition 4.1 *A double sequence $(A_{kl})_{k,l\in\mathbb{N}}$ of closed subsets of X is said to be Hausdorff convergent to a closed subset A of X if $\lim_{k,l\to\infty} h(A_{kl}, A) = 0$, in which case we write $H_2 - \lim_{k,l\to\infty} A_{kl} = A$.*

Definition 4.2 *Let $(A_{kl})_{k,l\in\mathbb{N}}$ be a double sequence of closed subsets of X. (A_{kl}) converges to A in the sense of Fisher if the following conditions hold:*

(α) : *For any $\varepsilon > 0$ there exists n_ε such that $\delta(A_{kl}, A) < \varepsilon$ for $k, l \geq n_\varepsilon$,*

(β) : *for any* $\varepsilon > 0$ *and* $x \in A$*, there exists* $n_{\varepsilon,x}$ *such that* $d(x, A_{kl}) < \varepsilon$ *for* $k, l \geq n_{\varepsilon,x}$.

In this case we write $F_2 - \lim_{k,l\to\infty} A_{kl} = A$.

Definition 4.3 *Let* $(A_{kl})_{k,l\in\mathbb{N}}$ *be a double sequence of subsets of a metric space* X*.* (A_{kl}) *converges to* A *in the sense of Wijsman if for any* $x \in X$ *we have* $\lim_{k,l\to\infty} d(x, A_{kl}) = d(x, A)$*. In this case we write* $W_2 - \lim_{k,l\to\infty} A_{kl} = A$.

Lemma 4.1 *Suppose that* $\{A; A_{kl}, k, l \in \mathbb{N}\}$ *is a family of closed subsets of* X*. Then* $H_2 - \lim_{k,l\to\infty} A_{kl} = A$ *if and only if either* A *and* A_{kl} *are empty for all* $k, l \geq n_0$ *or for any* $\varepsilon > 0$ *there exists* n_ε *such that for* $k, l \geq n_\varepsilon$,

$$A \subset A^{\varepsilon}_{kl} \quad \text{and} \quad A_{kl} \subset A^{\varepsilon}. \tag{4.1}$$

Proof. Note that $\lim_{k,l\to\infty} h(A_{kl}, A) = 0$ if and only if either A and A_{kl} are empty for $k, l \geq n_0$ or for all $\varepsilon > 0$, there exists n_ε such that for all $k, l \geq n_\varepsilon \Rightarrow h(A_{kl}, A) \leq \varepsilon$, or equivalently

$$\sup\{d(x, A) \mid x \in A_{kl}\} \leq \epsilon \quad \text{and} \quad \sup\{d(x, A_{kl}) \mid x \in A\} \leq \varepsilon.$$

This is exactly the meaning of 4.1. ■

Remark 4.2 *From Lemma 4.1,* $H_2 - \lim_{k,l\to\infty} A_{kl} = A$ *is equivalent to conditions* (α) *and* (γ) : *for any* $\varepsilon > 0$ *there exists* n_ε *such that* $\delta(A, A_{kl}) < \varepsilon$ *for* $k, l \geq n_\varepsilon$.

The following theorem exhibits the main relationship among these types of convergence.

Theorem 4.1 *For double sequences of closed sets we always have* $H_2 \Rightarrow F_2 \Rightarrow W_2 \Rightarrow K_2$.

Proof. $H_2 \Rightarrow F_2$. Since property (γ) implies property (β), the proof is obvious.

$F_2 \Rightarrow W_2$. Let $F_2 - \lim_{k,l\to\infty} A_{kl} = A$. If $A = \emptyset$, then there exists $n \in \mathbb{N}$ such that $A_{kl} = \emptyset$ for all $k, l \geq n$ and the implication is true. Now suppose that $A \neq \emptyset$. Given $\varepsilon > 0$ and $x \in X$. Then by condition (α) there exists n_ε such that $A_{kl} \subset A^\varepsilon$ for all $k, l \geq n_\varepsilon$. Thus, $d(x, A_{kl}) \geq d(x, A^\varepsilon)$. Now, it is not difficult to see that

$$d(x, A^\varepsilon) = \max\{0, d(x, A) - \varepsilon\}.$$

Therefore, $d(x, A) \leq d(x, A_{kl}) + \varepsilon$ for $k, l \geq n_\varepsilon$. This implies

$$d(x, A) \leq \liminf_{k,l\to\infty} d(x, A_{kl}). \tag{4.2}$$

To obtain the converse inequality, let $y \in A$ with $d(x, y) < d(x, A) + \varepsilon$. By condition (β) there exists $n_{\varepsilon,y}$ such that for $k, l \geq n_{\varepsilon,y}$ we have $d(y, A_{kl}) < \varepsilon$. Thus by (2.1),

$$d(x, A_{kl}) \leq d(x, y) + d(y, A_{kl}) < d(x, A) + 2\varepsilon.$$

This implies

$$\limsup_{k,l\to\infty} d(x, A_{kl}) \leq d(x, A) + 2\varepsilon.$$

Since ε is arbitrary

$$\limsup_{k,l\to\infty} d(x, A_{kl}) \leq d(x, A). \tag{4.3}$$

Combining (4.2) and (4.3), we have $\lim_{k,l\to\infty} d(x, A_{kl}) = d(x, A)$ which is desired.

$W_2 \Rightarrow K_2$. Let $W_2 - \lim_{k,l\to\infty} A_{k,l} = A$. If $A = \emptyset$, then $\lim_{k,l\to\infty} d(x, A_{kl}) = \infty$ for any x, which implies $\limsup_{k,l\to\infty} A_{kl} = \emptyset$, so $\lim_{k,l\to\infty} A_{kl} = A$. Now suppose that $A \neq \emptyset$; take $x \in A$, so

$$\lim_{k,l\to\infty} d(x, A_{kl}) = d(x, A) = 0.$$

This implies $x \in \liminf_{k,l\to\infty} A_{kl}$, so

$$A \subseteq \liminf_{k,l\to\infty} A_{kl}. \tag{4.4}$$

Now take $x \in \limsup_{k,l\to\infty} A_{kl}$. Then $\liminf_{k,l\to\infty} d(x, A_{kl}) = 0$. Since $W_2 - \lim_{k,l\to\infty} A_{kl} = A$, we get

$$d(x, A) = \lim_{k,l\to\infty} d(x, A_{kl}) = 0.$$

Thus, $x \in A$. This means that

$$\limsup_{k,l\to\infty} A_{kl} \subseteq A. \tag{4.5}$$

By inclusions (4.4) and (4.5) we obtain $\lim_{k,l\to\infty} A_{kl} = A$. ■

Now we give some examples which show that the converses of the implications in Theorem 4.1 are not true in general.

Example 4.1 *Let $X = l^2$. Define the double sequence of sets*

$$A_{kl} := [e_1, e_{\max\{k,l\}}].$$

Then $\lim_{k,l\to\infty} A_{kl} = A = \{e_1\}$ *but* $W_2 - \lim_{k,l\to\infty} A_{kl} \neq A$, *since* $d(\theta, A_{kl}) = \frac{1}{\sqrt{2}}$ *and* $d(\theta, A) = 1$.

Example 4.2 *Let $X = \mathbb{R}^2$. Take the double sequence of sets*

$$A_{kl} := \left\{(x,y) : 0 \leq x \leq k \cdot l; \quad 0 \leq y \leq \frac{1}{k \cdot l} x\right\}.$$

We have $W_2 - \lim_{k,l\to\infty} A_{kl} = A$, where $A = \{(x,y) : 0 \leq x; y = 0\}$ but $F_2 - \lim_{k,l\to\infty} A_{kl} \neq A$.

Example 4.3 *Let $X = \mathbb{R}$. Take the double sequence of sets*

$$A_{kl} := [-(k+l), (k+l)].$$

We have $F_2 - \lim_{k,l\to\infty} A_{kl} = \mathbb{R}$ but $H_2 - \lim_{k,l\to\infty} A_{kl} \neq \mathbb{R}$.

Lemma 4.3 *Let (X,d) be a metric space and K be a compact subset of X. Then, we have $K \cap P_x \neq \emptyset$ for every double sequence (x_{kl}) with $\{(k,l) : x_{kl} \in K\} \in \mathcal{N}_2^{\#}$.*

Proof. Since $M = \{(k,l) : x_{kl} \in K\} \in \mathcal{N}_2^{\#}$, let us denote the first terms of elements of M by k_i and the second ones by l_i. Then (k_i) and (l_i) are increasing sequences. Let us define $y_i = x_{k_i l_i}$ for all $i \in \mathbb{N}$. Since, the sequence (y_i) belongs to the compact set K, there exists a subsequence (y_{i_n}) such that

$$\lim_{n\to\infty} y_{i_n} = \lim_{n\to\infty} x_{k_{i_n} l_{i_n}} = y_0 \in K.$$

It is trivial that y_0 is a Pringsheim limit point of the sequence x. Hence $K \cap P_x \neq \emptyset$. ■

Definition 4.4 *The double sequence (A_{kl}) is said to be Pringsheim bounded if there exists a compact set K and $n \in \mathbb{N}$ such that $A_{kl} \subseteq K$ for all $k,l \geq n$.*

The next theorem shows that for a Pringsheim bounded double sequence of closed sets the types of convergence mentioned above are equivalent.

Theorem 4.2 *Let (A_{kl}) be a Pringsheim bounded double sequence of closed subsets of X. If $\lim_{k,l} A_{kl} = A$ with $A \neq \emptyset$, then $H_2 - \lim A_{kl} = A$.*

Proof. $\lim_{k,l} A_{kl} = A$. Hence, the closed set A is compact. Then given $\varepsilon > 0$, A has a finite cover with open balls of radius ε, i.e., there exists $\{x_1, x_2, x_3, \ldots, x_n\}$ with $x_i \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B\left(x_i, \frac{\varepsilon}{2}\right).$$

Since $A \neq \emptyset$ and $x_i \in A$ for $i \in \{1, 2, \ldots, n\}$, we obtain $\lim_{k,l\to\infty} d(x_i, A_{kl}) = 0$. Therefore, there exists $n \in \mathbb{N}$ such that $d(x_i, A_{kl}) < \frac{\varepsilon}{2}$ for $k, l \geq n$ and each i. Thus, for any $y \in A$ we obtain

$$d(y, A_{kl}) \leq d(y, x_i) + d(x_i, A_{kl}) < \varepsilon.$$

Hence, $A \subseteq A^{\varepsilon}_{kl}$ for every $k, l \geq n$.

Now, suppose that there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ we have $A_{kl} \nsubseteq A^{\varepsilon}$ for some $k, l \geq n$. That is,

$$M = \big\{(k,l) : A_{kl} \nsubseteq A^{\varepsilon}\big\} \in \mathcal{N}_2^{\#}.$$

Hence, there exists a sequence $\{y_{kl}, (k,l) \in M \mid y_{kl} \in A_{kl} \backslash A^{\varepsilon}\} \subseteq K$. By Lemma 4.3, the sequence (y_{kl}) has at least Pringsheim limit point that belongs to $\limsup_{k,l\to\infty} A_{kl} = A$ but does not belong to $A^{\varepsilon} \supseteq A$. This is a contradiction, so there exists $n \in \mathbb{N}$ such that $A \subseteq A^{\varepsilon}_{kl}$ for every $k, l \geq n$, which completes the proof. ■

The following result is analogue of Lemma 3.4 due to Beer [6].

Theorem 4.3 *Let (X, d) be a metric space and let (A_{kl}) be a double sequence of nonempty closed subsets. Suppose $(d(., A_{kl}))$ is pointwise convergent to a finite-valued function. Then (A_{kl}) is Kuratowski convergent.*

Proof. In order to prove the theorem, we only need to show $\limsup_{k,l\to\infty} A_{kl} \subseteq \liminf_{k,l\to\infty} A_{kl}$. Take an arbitrary $x \in \limsup_{k,l\to\infty} A_{kl}$ and let $\varepsilon > 0$. By Cauchyness of $(d(x, A_{kl}))$, choose $N \in \mathbb{N}$ such that

$$k, l, p, q \geq N \Rightarrow |d(x, A_{kl}) - d(x, A_{pq})| < \frac{\varepsilon}{2}.$$

Then choose $r, s > N$ such that $B(x, \frac{\varepsilon}{2}) \cap A_{rs} \neq \emptyset$. For these r, s we have

$$d(x, A_{kl}) \leq d(x, A_{rs}) + |d(x, A_{kl}) - d(x, A_{rs})| < \varepsilon.$$

This means $\forall k, l \geq N$, we have $B(x, \varepsilon) \cap A_{kl} \neq \emptyset$. By definition we get $x \in \liminf_{k,l\to\infty} A_{kl}$ and this step completes the proof. ■

The following theorem shows that in normed linear spaces Wijsman convergence of double sequences of closed sets can be expressed in terms of Kuratowski convergence of closed enlargements. This result was obtained by Dolecki [9] for single sequences of sets.

Theorem 4.4 *Let $(X, \|\cdot\|)$ be a normed linear space. Let (A_{kl}) be a double sequence of closed subsets of X, and let A be a closed set. Then (A_{kl}) is Wijsman convergent to A*

if and only if for each $\varepsilon > 0$, the double sequence of ε-enlargements $\overline{A_{kl}^{\varepsilon}}$ is Kuratowski convergent to $\overline{A^{\varepsilon}}$.

Proof. Sufficiency holds in an arbitrary metric space. First, suppose that $x_0 \in X$ is fixed and $d(x_0, A) < \varepsilon$. Choose a scalar β with $d(x_0, A) < \beta < \varepsilon$. Clearly, $x_0 \in \overline{A^{\beta}}$ and by assumption,

$$\overline{A^{\beta}} \subseteq \liminf_{k,l\to\infty} \overline{A_{kl}^{\beta}}.$$

Thus, there exists $n \in \mathbb{N}$ such that for each $k, l \geq n$, we have

$$B(x_0, \varepsilon - \beta) \cap \overline{A_{kl}^{\beta}} \neq \emptyset.$$

Therefore we have $d(x_0, A_{kl}) < \varepsilon$ for each $k, l \geq n$. On the other hand, suppose that for each $n \in \mathbb{N}$ there exist $k, l \geq n$ such that $d(x_0, A_{kl}) \leq \varepsilon$. That is, $x_0 \in \overline{A_{kl}^{\varepsilon}}$. From that it follows

$$x_0 \in \limsup_{k,l\to\infty} \overline{A_{kl}^{\varepsilon}} \subseteq \overline{A^{\varepsilon}}$$

in which case $d(x_0, A) \leq \varepsilon$. Thus, $d(x_0, A) > \varepsilon$ ensures that there exists $n \in \mathbb{N}$ such that for each $k, l \geq n$, we have $d(x_0, A_{kl}) > \varepsilon$.

For necessity, fix $\varepsilon > 0$. We must show that

$$\overline{A^{\varepsilon}} \subseteq \liminf_{k,l\to\infty} \overline{A_{kl}^{\varepsilon}} \quad \text{and} \quad \limsup_{k,l\to\infty} \overline{A_{kl}^{\varepsilon}} \subseteq \overline{A^{\varepsilon}}.$$

For the first inclusion, fix $x_0 \in \overline{A^{\varepsilon}}$ and let $\delta > 0$. Choose $a \in A$ with

$$\|x_0 - a\| < \varepsilon + \frac{\delta}{2}.$$

By Wijsman convergence, $\lim_{k,l\to\infty} d(a, A_{kl}) = d(a, A) = 0$. Therefore, there exists $n \in \mathbb{N}$ such that for each $k, l \geq n$, we have $d(a, A_{kl}) < \frac{\delta}{2}$. Then for each $k, l \geq n$, there exists $a_{kl} \in A_{kl}$ such that

$$\|a_{kl} - a\| < \frac{\delta}{2}.$$

Then $\|x_0 - a_{kl}\| < \varepsilon + \delta$ and so the line segment joining x_0 to a_{kl} contains a point of $B(x_0, \delta) \cap \overline{A_{kl}^{\varepsilon}}$ for $k, l \geq n$ and the inclusion

$$\overline{A^{\varepsilon}} \subseteq \liminf_{k,l\to\infty} \overline{A_{kl}^{\varepsilon}}$$

follows. For the second inclusion, let $x_0 \in \limsup_{k,l\to\infty} \overline{A_{kl}^{\varepsilon}}$ be arbitrary. For each $\delta > 0$ and $n \in \mathbb{N}$ there exist $k, l \geq n$ such that

$$B(x_0, \delta) \cap \overline{A_{kl}^{\varepsilon}} \neq \emptyset$$

and so $d(x_0, A_{kl}) < \varepsilon + \delta$. By Wijsman convergence, we get

$$d(x_0, A) = \liminf_{k,l\to\infty} d(x_0, A_{kl}) \leq \varepsilon.$$

This means that $x_0 \in \overline{A^\varepsilon}$, and so $\limsup_{k,l\to\infty} \overline{A^\varepsilon_{kl}} \subseteq \overline{A^\varepsilon}$, as required. ■

5 Monotone double sequences

In this section, we introduce monotone double sequences and examine limit of monotone double sequences of sets.

Definition 5.1 *$(A_{kl})_{k,l\in\mathbb{N}}$ is called an increasing double sequence of sets if the following conditions hold:*

(i) For each $k \in \mathbb{N}$ and for every $l \in \mathbb{N}$, $A_{kl} \subseteq A_{k,l+1}$,

(ii) for each $l \in \mathbb{N}$ and for every $k \in \mathbb{N}$, $A_{kl} \subseteq A_{k+1,l}$.

Theorem 5.1 *Suppose that $(A_{kl})_{k,l\in\mathbb{N}}$ is an increasing double sequence of closed subsets of X. Then $\lim_{k,l} A_{kl}$ exists and*

$$\lim_{k,l\to\infty} A_{kl} = cl \bigcup_{k,l\in\mathbb{N}} A_{kl}.$$

Proof. Let $A = cl \bigcup_{k,l\in\mathbb{N}} A_{kl}$. Clearly $A_{kl} \subset A$ for all $k, l \in \mathbb{N}$. Thus if A is empty, it follows that A_{kl} is empty for all $k, l \in \mathbb{N}$ and the theorem holds trivially.

Now we assume that A is nonempty and take x in A. In this case, for every $\varepsilon > 0$

$$B(x,\varepsilon) \cap \left(\bigcup_{k,l\in\mathbb{N}} A_{kl} \right) \neq \emptyset.$$

Then there exist $k_1, l_1 \in \mathbb{N}$ such that $B(x,\varepsilon) \cap A_{k_1 l_1} \neq \emptyset$.

Let us define $n = \max\{k_1, l_1\}$. Since $(A_{kl})_{k,l\in\mathbb{N}}$ is an increasing double sequence, for all $k, l \geq n$, $A_{k_1 l_1} \subseteq A_{kl}$. Hence, $B(x,\varepsilon) \cap A_{kl} \neq \emptyset$ for all $k, l \geq n$. This means that $x \in \liminf_{k,l\to\infty} A_{kl}$.

It remains to show that $\limsup_{k,l\to\infty} A_{kl} \subset A$. Let $x \in \limsup_{k,l\to\infty} A_{kl}$ be arbitrary. Then for every $\varepsilon > 0$ and every $n \in \mathbb{N}$ there exist $k, l \geq n$ such that $B(x,\varepsilon) \cap A_{kl} \neq \emptyset$. It follows that

$$B(x,\varepsilon) \cap \left(\bigcup_{k,l\in\mathbb{N}} A_{kl} \right) \neq \emptyset,$$

and thus $x \in cl \bigcup_{k,l} A_{kl} = A$. This completes the proof. ■

Remark 5.1 *(β) property in the definition of F_2 convergence of (A_{kl}) to A means that $\lim_{k,l\to\infty} d(x, A_{kl}) = 0$ for every $x \in A$, i.e.,*

$$A \subseteq \liminf_{k,l\to\infty} A_{kl}.$$

Therefore, we obtain the following equivalence: In the case when (α) holds, $F_2 - \lim A_{kl} = A$ is equivalent to $\lim_{k,l\to\infty} A_{kl} = A$. For an increasing double sequence $(A_{kl})_{k,l\in\mathbb{N}}$ we have (α), so

$$F_2 - \lim_{k,l\to\infty} A_{kl} = cl \bigcup_{k,l\in\mathbb{N}} A_{kl}.$$

In general, we do not have

$$H_2 - \lim_{k,l\to\infty} A_{kl} = cl \bigcup_{k,l\in\mathbb{N}} A_{kl},$$

as shown by Example 4.3.

Definition 5.2 *$(A_{kl})_{k,l\in\mathbb{N}}$ is called a decreasing double sequence if the following conditions hold:*

(i) For each $k \in \mathbb{N}$ and for every $l \in \mathbb{N}$, $\quad A_{kl} \supseteq A_{k,l+1}$,

(ii) for each $l \in \mathbb{N}$ and for every $k \in \mathbb{N}$, $\quad A_{kl} \supseteq A_{k+1,l}$.

Theorem 5.2 *Suppose that $(A_{kl})_{k,l\in\mathbb{N}}$ is a decreasing double sequence of closed subsets of X. Then $\lim_{k,l\to\infty} A_{kl}$ exists and*

$$\lim_{k,l\to\infty} A_{kl} = \bigcap_{k,l\in\mathbb{N}} A_{kl}.$$

Proof. Let $A = \bigcap_{k,l\in\mathbb{N}} A_{kl}$. Clearly if $x \in A$, then for every $k, l \in \mathbb{N}$, $x \in A_{kl}$ and $B(x, \varepsilon) \cap A_{kl} \neq \emptyset$. This means that $x \in \liminf_{k,l\to\infty} A_{kl}$.

It remains to show that $\limsup_{k,l\to\infty} A_{kl} \subseteq A$. Let $x \in \limsup_{k,l\to\infty} A_{kl}$ be arbitrary. Then for every $k, l \in \mathbb{N}$ and for every $\varepsilon > 0$ there exist $k_1, l_1 \geq \max\{k, l\}$ such that $B(x, \varepsilon) \cap A_{k_1 l_1} \neq \emptyset$.

Since $(A_{kl})_{k,l\in\mathbb{N}}$ is a decreasing double sequence, $A_{k_1 l_1} \subseteq A_{kl}$. Hence, $B(x, \varepsilon) \cap A_{kl} \neq \emptyset$ for all $k, l \in \mathbb{N}$. This means that $x \in clA_{kl}$. Since A_{kl} is closed, $x \in A_{kl}$. Therefore, $x \in \bigcap_{k,l\in\mathbb{N}} A_{kl}$. This completes the proof. ■

Example 5.1 *Let $X = l^2$. Define the decreasing double sequence of sets*

$$A_{kl} =: \{e_{k+l}, e_{k+l+1}, e_{k+l+2}, ...\}.$$

Then $\bigcap_{k,l\in\mathbb{N}} A_{kl} = \emptyset$. *However, if*

$$x = \left(\frac{1}{2^{1/2}}, \frac{1}{2^{2/2}}, \frac{1}{2^{3/2}}, \dots\right),$$

then for each $n \in \mathbb{N}$ *we have*

$$\|e_n - x\| = \sqrt{2\left(1 - \frac{1}{2^{n/2}}\right)}.$$

Hence,

$$d(x, A_{kl}) = \inf_{n\geq k+l} \|e_n - x\| = \|e_{k+l} - x\| = \sqrt{2\left(1 - \frac{1}{2^{(k+l)/2}}\right)}.$$

Therefore, $\lim_{k,l\to\infty} d(x, A_{kl}) = \sqrt{2} \neq d(x, \emptyset)$. *This means that* $W_2 - \lim_{k,l\to\infty} A_{kl} \neq \emptyset$.

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